Exercises on General Relativity TVI TMP-TC1

Problem set 8, due December 18th

Exercise 1 – Isomorphism between Vectors and 1-forms

A symmetric, non-degenerate bilinear form on the tangent space of a manifold is called a metric. For a given metric g and a vector u, we can define a 1-form ω_u by its action on an arbitrary vector v:

$$\omega_u(v) = g(u, v) \ . \tag{1}$$

Consider a coordinate basis of vectors $\frac{\partial}{\partial x^\mu}$ and 1-forms $\mathrm{d}x^\mu$. Show that the components of the 1-form ω_v are given by the formula

$$(\omega_u)_\mu = g_{\mu\nu} u^\nu \ , \tag{2}$$

where $g_{\mu\nu}$ are the components of the metric in a given coordinate system.

Exercise 2 – Vector Equations and Ricci Calculus

Solve for the unknown vector X^α in the following equations:

$$\text{i)} \quad kX^\alpha + \epsilon^{\alpha\beta\gamma} X_\beta A_\gamma = B^\alpha \qquad (\text{in } \mathbb{R}^3) \tag{3}$$

$$\text{ii)} \quad X^\alpha A_\alpha = k \quad \text{and} \quad X^\alpha B_\alpha = l \qquad (\text{in } \mathbb{R}^2) \ , \tag{4}$$

where k and l are non-zero scalars, A and B are linearly independent vectors and ϵ is the Levi-Civita tensor.

Exercise 3 – Commutative properties of vector fields

Consider the Lie bracket of two smooth vector fields X and Y on a manifold M:

$$[X, Y](f) := XY(f) - YX(f) \,, \tag{5}$$

where $f \in C^\infty(M)$.

(i) Show that the vector fields fulfil the Jacobi identity:

$$\big[[X, Y], Z\big] + \big[[Z, X], Y\big] + \big[[Y, Z], X\big] = 0 \tag{6}$$

(ii) Furthermore,

$$[fX, Y] = f[X, Y] - Y(f)X \ . \tag{7}$$

(iii) Let from now on $M = \mathbb{R}^n$. Show that the bracket of the coordinate vector fields $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \in \mathrm{Vec}(\mathbb{R}^n)$ vanishes.

(iv) Show that, for $X, Y \in \mathrm{Vec}(\mathbb{R}^n)$, if

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j} \tag{8}$$

then the bracket is given by

$$[X, Y] = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left(f_i \frac{\partial g_j}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i} \right) \right) \frac{\partial}{\partial x_j} \ . \tag{9}$$

(v) As an example calculate the bracket of the vector fields $X, Y \in \mathrm{Vec}(\mathbb{R}^2 \setminus \{0\})$ with

$$X = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \ , \tag{10}$$

where $r := \sqrt{x^2 + y^2}$.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`