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T VI: Nonlinear Dynamics and Complex Systems
(Prof. E. Frey)

Problem Set 8:
Linear Stability analysis and stripe patterns

Problem 1 *Hydrodynamics of propelled particles*

Starting from a mesocopic Boltzmann equation hydrodynamic equations for a fluid consisting of propelled particles can be derived (PRE 77, 022101, 2006). This approach stems from the use of an explicit alignment rule, e.g. two particles with an initial relative angle collide within a certain interaction distance such that the new relative angle reduces. In comparision to the model discussed in the last problem set, this model does not incoperate the impact of hard-core interactions. Only the impact of noise enters in the value of the kinetic coefficients, which should not be discussed here. However, on the level of the hydrodynamic equations the propelled particles are descibed by a concentration field $c(\vec{x}, t)$ and a polarization density field $\vec{u}(\vec{x}, t) = c(\vec{x}, t)\vec{P}(\vec{x}, t)$, where $\vec{P}(\vec{x}, t)$ denotes the polarization field. The full equations derived first from a Boltzmann equation by Gregoire et.al. are:

$$\partial_t c + v_0 \nabla \cdot \vec{u} = 0, \tag{1}$$

$$\partial_t \vec{u} + \gamma(\vec{u} \cdot \nabla)\vec{u} + \kappa(\nabla \cdot \vec{u})\vec{u} - \frac{\kappa}{2}\nabla \vec{u}^2 + \frac{1}{2}v_0 \nabla c = \alpha \vec{u} - \beta \vec{u}^2 \vec{u} + \eta \nabla^2 \vec{u} + \xi \nabla\nabla \cdot \vec{u}. \tag{2}$$

where the kinetic coefficients γ, κ, β, η and ξ are positive. The constant α can change the sign depending on the mean density, as we will see later. The Boltzmann approach gives the functional form of the kinetic coefficients on the parameters of the collision rule as well their density dependence. The later should be ignored for instance.

a) Recall the Navier-Stokes equations describing the motion of a compressible fluid (e.g. a gas). Explain the physical meaning of each term. Argue the meaning of the new terms in comparision to a Navier-Stokes fluid, which arise in the hydrodynamic description for a fluid which consists of propelled particles!

b) Assume α to be of the functional form: $\alpha = \alpha_0(c - c_c)/c$. Consider the spatially homogeneous system ($c = const.$). Discuss of the homogeneous unpolarized and polarized state! For which density is the polarized state expected to be stable?

c) Perform a linear stability analysis for the unpolarized ($\vec{u} = 0$), homogeneous distributed state ($c = c_0$) for $c_0 < c_c$ and show that it is stable for all wavenumbers. For simplicity consider pertubations in the polarisation parallel to the wavevector $\vec{k}$ only (see problem 2, problem set 7)!

d) Now investigate a linear stability for the homogeneous concentrated, polarized state: $c(\vec{x},t) = c_0 + \delta c(\vec{x},t)$ and $\vec{u}(\vec{x},t) = \vec{u}_0 + \delta\vec{u}(\vec{x},t)$. Test the stability for $c_0 > c_c$. Sketch the real part of the larger eigenvalue by arguing the asymptotics in k (instead of a tremendous algebra problem). Discuss how to decrease the unstable waveband!
Hint: use $\sqrt{x+\mathrm{i}y} = \sqrt{(r+x)/2} + \mathrm{i}\frac{y}{\sqrt{2(r+x)}}$ with $r = \sqrt{x^2+y^2}$. For simplicity consider pertubations in the polarisation parallel to the wavevector.

Note: The stability analysis for arbitrary directed perturbations is a necessary task but one ends up with a tremendous algebraic effort without essential insights, so we skip this here.

e) From a stability analysis on the level of the Boltzmann equation one can conclude that there is a region where the polarized state state is only stable for spacially inhomogeneous distributions. This motivates to look for travelling wave solutions! Therefore apply the following ansatz: $c(\vec{x},t) = R(x-ct)$ and $\vec{u}(\vec{x},t) = W(x-ct)\vec{e}$. Use the continuity equation to substitute R in favor of W to end with an ODE of second order for W. Determine the corresponding potential (for the rolling ball analogy). Is the profile for u symmetric?

f) * Integrate the non-linear wave-ODE numerically.

Problem 2 *Stripe patterns in the Swift-Hohenberg equation*

The Swift-Hohenberg equation discussed in the lecture represents a simple model of a pattern forming system whose uniform base state $u = 0$ undergoes a I-s-instability. In one spacial dimension the Swift-Hohenberg equation has the form:

$$\partial_t u = (r-1)u - 2\partial_x^2 u - \partial_x^4 u - u^3. \tag{3}$$

When performing a stability analysis one finds that the uniform state $u = 0$ becomes linearily unstable at $r = 0$ (as the control parameter r is increased from negative values), leading to an exponentially growing mode which can be written as:

$$u \propto \exp(rt)\cos(q_c x)$$

with the critical wavenumber (the first unstable mode) $q_c = 1$. The aim is to understand the non-linear saturation of this growing mode for small r. In detail, we want to work out the stationary spacial structure of the resulting pattern, the stripe. For simplicity we will restrict to one wavenumber, the critical wavenumber $q_c = 1$, which is reasonable because it is the most unstable mode of the appearing unstable waveband.

a) Argue that the following form is a natural guess for the stationary stripe pattern:

$$u(x) = a_1\cos(x).$$

Show that it is a solution. Calculate a_1! Find the contradiction and argue to change the ansatz to:

$$u(x) = a_1\cos(x) + a_3\cos(3x).$$

b) Use the refined ansatz to derive a hierachy of the coefficents a_1, a_3, ... as a function of the parameter r.

c) Finally, write down the spacial distribution of $u(x)$ in leading orders in r, proving the emergence of a stripe state in the Swift-Hohenberg equation.