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Prof. Dr. E. Frey

Florian Thüroff und Christoph Weber

Lehrstuhl für Statistische Physik
Biologische Physik & Weiche Materie
Arnold-Sommerfeld-Zentrum für Theoretische Physik
Department für Physik

Ludwig Maximilians Universität

T VI: Nonlinear Dynamics and Complex Systems

(Prof. E. Frey)

Amplitude Equations

Problem1 *Amplitude equation for the Lorenz equation*

The Lorenz equations model the Rayleigh-Bernard convection, with X representing the circulation velocity in the convection, Y the spatially periodic temperature perturbation and Z the heat transport due to the convection.

$$\begin{aligned} \frac{dX}{dt} &= -\sigma(X-Y), \\ \frac{dY}{dt} &= rX - Y - XZ, \\ \frac{dZ}{dt} &= b(XY - Z), \end{aligned} \tag{1}$$

However, the physical meaning is negligible for the requested derivation of the amplitude equation in this problem.

a) Perform a linear stability analysis of Eqs. 1 and show that the uniform solution $X = Y = Z = 0$ undergoes a bifurcation at the critical value $r_c = 1$.

b) To investigate the amplitude of the observables $\vec{u}_p = (X, Y, Z)$ slightly above the critical point, we expand $\vec{u}_p$ around the uniform state $\vec{u} = 0$:

$$\vec{u}_p = \varepsilon^{1/2}\vec{u}_0 + \varepsilon\vec{u}_1 + \dots,$$

where $\varepsilon = r - r_c$ is the small deviation from the critical point $r_c = 1$. Write down the evolution equation of $\vec{u}_p$ as:

$$L\vec{u}_p = \frac{d\vec{u}_p}{dt} - N[\vec{u}_p], \tag{2}$$

where L denotes the linear operator and N the remaining nonlinear operator of the system in Eqs. 1.

c) Expand the linear operator $L = L_0 + \varepsilon^{1/2}L_1 + \varepsilon L_2$ and investigate the slow time behavior of Eq. 2 by setting $T = \varepsilon t$. Find the equations determining $\vec{u}_0$, $\vec{u}_1$ by comparison of coefficients of different orders of ε.

d) By calculating the eigenvalues and eigenvectors of L_0 derive $\vec{u}_0$ and $\vec{u}_1$. Note that the zero eigenvalue corresponds to the onset of the linear instability and hence constitutes the first order.

e) By ordering the coefficients of $\varepsilon^{(3/2)}$ find the equation for the first order amplitude from the requirement of no resonant terms. For this calculation note the following Theorem:

Theorem (Fredholm alternative)
The vector $\vec{u}$ is a solution of $L\vec{u} = \vec{b}$ provided the right hand side $\vec{b}$ is orthogonal to the null eigenvector of the adjoint operator $L^\dagger$.

Problem2 *Eckhaus instability*

Consider a physical system exhibiting a simple stripe state in the vicinity of a type-I_s instability, occurring at some critical parameter value p_c. Assuming the system to be translationally invariant and symmetric under space inversion, the amplitude equation reads

$$\tau_0 \partial_t A = \varepsilon A + \xi_0^2 \partial_x^2 A - g_0 |A|^2 A. \tag{3}$$

Here $\varepsilon \equiv (p - p_c)/p_c \ll 1$ quantifies the distance from criticality and τ_0, ξ_0 and g_0 depend on the details of the particular system under study.

a) Employ the amplitude equation (3) to confirm that the possible wavenumbers q, characterizing the stripe pattern, are constrained to lie within the existence band $q \in E = \{q_c - \varepsilon^{1/2}/\xi_0, q_c + \varepsilon^{1/2}/\xi_0\}$, whose width scales $\Delta q \sim \varepsilon^{1/2}$.

b) From now on focus on a particular stripe pattern of wavenumber $q = q_c + k$ ($q \in E$) and test its stability against small perturbations $\delta A(x, t)$. To this end show that the amplitude equation (3) can be linearized in the small perturbation $\delta A(x, t)$ to give

$$\tau_0 \partial_t \delta A = \varepsilon \delta A - \xi_0^2 \partial_x^2 \delta A - g_0 \left(A^2 \delta A^* + 2|A|^2 \delta A \right), \tag{4}$$

where $A = a_k e^{ikx}$ denotes the amplitude of the unperturbed stripe pattern.

c) Employ the general ansatz

$$\delta A(x, t) = e^{ikx} \left[\delta a_+(t) e^{iQx} + \delta a_-^*(t) e^{-iQx} \right],$$

to determine the growth rate $\sigma_k(Q)$ from Eq. (4). Recall that the growth rate is implicitly defined via $\delta a_\pm \sim e^{\sigma_k(Q)t}$ (it is sufficient to consider only the largest growth rate).

d) Modes for which $\max_Q \sigma_k(Q) > 0$ holds true become unstable—the Eckhaus instability. By inspecting the small wavenumber limit $Q \to 0$ of $\sigma_k(Q)$, show that the Eckhaus instability itself is of type-II and that it shrinks the width of the band of stable modes by a universal factor of $1/\sqrt{3}$ compared to the width of the existence band E.

Problem3 *Amplitude equations for 1D systems.*

a) Using symmetry arguments derive the most general amplitude equation for the state arising from an instability at critical wave number q_c and critical control parameter p_c, where the instability is stationary in a one-dimensional system that does not have parity symmetry (i.e. is not invariant under $x \to -x$) but is invariant under change of sign of the field $\mathbf{u} \to -\mathbf{u}$. What is the nature of the nonlinear state slightly above onset and for wave numbers slightly away form q_c?

b) Compute analytically the shape and speed of the moving front connecting the saturated stripe solution with the uniform state for the amplitude equation (3).