Perturbation theory for anharmonic oscillations

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In these notes I explain the basics of perturbation theory and then show how to apply this theory to describe anharmonic oscillations.

1 Introduction

Consider a one-dimensional system described by a coordinate $x(t)$. This corresponds to a point mass moving in a potential $V(x)$. When the system is near a stable equilibrium point $x = x_0$, we have $V'(x_0) = 0$, and thus the equation of motion

$$\ddot{x} = -\frac{1}{m}V'(x)$$

can be linearized as follows,

$$\ddot{x} = -(x - x_0)\frac{V''(x_0)}{m} - \frac{1}{2}(x - x_0)^2\frac{V'''(x_0)}{m} - ... \qquad (1)$$

If we only consider the linear term (**harmonic approximation**), we obtain the equation of a harmonic oscillator with the frequency

$$\omega^2 \equiv \frac{1}{m}V''(x_0).$$

Harmonic oscillations have a fixed period, $T = 2\pi\omega^{-1}$, which is independent of the amplitude. However, this is only an approximation which is valid for small enough $|x - x_0|$. It is also important to study the effect of further terms in Eq. (1). For instance, we would like to know how the period of oscillations depends on the amplitude when the amplitude becomes large. This will give us an idea about the limits of applicability of the harmonic approximation.

Let us introduce the coordinate $q(t) \equiv x(t) - x_0$. We are interested in describing an **anharmonic oscillator**,

$$\ddot{q} + \omega_0^2 q = f(q), \qquad (2)$$

where $f(q)$ is a nonlinear function which represents a small perturbation. The function $f(q)$ can be expanded in a power series in q as follows,

$$f(q) = f_2 q^2 + f_3 q^3 + ...,$$

where the parameters f_2, f_3, ..., are "small" in an appropriate sense. (Below we shall determine exactly how small they must be.)

In the general case, it is impossible to obtain an exact solution of Eq. (2). However, one can use the method called "perturbation theory" and obtain an *approximate* solution as a function of the small parameters f_2, f_3, ...

I now review the basic principles of perturbation theory. (Please follow all the calculations with pen and paper if this is your first experience with perturbation theory.)

2 What is perturbation theory

Often we have a problem whose exact solution cannot be found, but a slightly different problem is exactly solvable. Perturbation theory is a method for obtaining an *approximate* solution of the first problem by modifying the *exact* solution of the second problem.

2.1 A first example

Consider an **algebraic** (not differential) equation,

$$x^{5.23} - 0.000001\,x - 1 = 0. \qquad (3)$$

It is clear that this equation cannot be solved exactly. But we notice that Eq. (3) is almost the same as the equation

$$x^{5.23} - 1 = 0,$$

which we can easily solve: $x = 1$. In such situations, one can apply perturbation theory with great success.

The method of perturbation theory is applied to Eq. (3) as follows. For brevity, let us set $a \equiv 5.23$. The first step is to introduce a *small parameter* into the equation. One says

that Eq. (3) is a **perturbation** of the equation $x^a - 1 = 0$, and one describes this perturbation by introducing a small parameter ε, so that Eq. (3) becomes

$$x^a - \varepsilon x - 1 = 0. \tag{4}$$

In the particular example, we have $\varepsilon = 10^{-6}$, but it is actually easier to keep the value of ε arbitrary, as long as we remember that it is very small. The value $\varepsilon = 0$ describes the **unperturbed equation** which we can solve exactly; the **unperturbed solution** is $x_0 = 1$. Now we are ready for the second step. It is reasonable to suppose that the solution x_* of Eq. (4) is a small perturbation of the unperturbed solution, i.e. that x_* is only very slightly different from 1. We may imagine that we could solve Eq. (4) for every (sufficiently small) ε, so that the solution x_* is a function of ε. Since we know that $x_*(\varepsilon = 0) = x_0 = 1$, it seems reasonable to *assume* that we may describe the solution $x_*(\varepsilon)$ as a series in ε,

$$x_*(\varepsilon) = 1 + A_1\varepsilon + A_2\varepsilon^2 + ..., \tag{5}$$

where $A_1, A_2, ...$ are some unknown constants. I would like to emphasize that the formula (5) is just a guess; at this point, we have no proof that the solution x_* is indeed of this form. We need to check that this guess is correct. Let us substitute Eq. (5) into Eq. (4) and expand in powers of ε: the constants $A_1, A_2, ...$ must be such that

$$\left(1 + A_1\varepsilon + A_2\varepsilon^2 + ...\right)^a - \varepsilon\left(1 + A_1\varepsilon + A_2\varepsilon^2 + ...\right) - 1 = 0. \tag{6}$$

For the moment, let us keep only linear terms in ε:

$$1 + aA_1\varepsilon - \varepsilon - 1 + O(\varepsilon^2) = 0.$$

Since ε is arbitrary, the linear terms in ε must cancel separately from the quadratic terms, so we must have

$$A_1 = \frac{1}{a}.$$

Now let us look at the quadratic terms in Eq. (6) and disregard the cubic terms. We find

$$aA_2\varepsilon^2 + \frac{a(a-1)}{2}A_1^2\varepsilon^2 - A_1\varepsilon^2 = 0, \tag{7}$$

therefore

$$A_2 = -\frac{1}{2a} + \frac{3}{2a^2}.$$

Let us stop at this point and summarize what we have done. Using perturbation theory, we have found an *approximate* solution of Eq. (4),

$$x_* \approx 1 + A_1\varepsilon + A_2\varepsilon^2 = 1 + \frac{1}{a}\varepsilon - \frac{a-3}{2a^2}\varepsilon^2. \tag{8}$$

For the values $\varepsilon = 10^{-6}$ and $a = 5.23$, the term $A_2\varepsilon^2$ is only a tiny correction of order 10^{-13}, so the precision obtained by using only the linear term, $x_* \approx 1 + \varepsilon/a$, is already sufficient for almost all practical purposes.

I would like to contrast the result obtained using perturbation theory, Eq. (8), with a **numerical** solution (i.e. a solution found by a **numerical method**, which essentially consists of substituting different numbers x_* into Eq. (3) until a certain precision is achieved). By using a computer program, one could easily obtain an approximate solution of Eq. (3) with particular numerical coefficients $\varepsilon = 10^{-6}, a = 5.23$. However, one would obtain a *single* number x_*; it would remain unclear how this solution depends on a and ε. The numerical procedure would have to be repeated if a different value of a or ε were given. The perturbative result (8) is much more useful because it shows the behavior of x_* as a function of ε and a.

2.2 Limits of applicability of perturbation theory

The solution (8) is not exact but *approximate*; what is the precision of this approximation? Here is how one can get a qualitative answer without knowing the exact solution. The second term in Eq. (8) is linear in ε and represents a small correction to the first term if $|\varepsilon| \ll a$. The third term represents a small correction to the second term if

$$\frac{|a-3|}{2a^2}\varepsilon^2 \ll \frac{|\varepsilon|}{a},$$

which is also satisfied if $|\varepsilon| \ll a$. Therefore, the condition $|\varepsilon| \ll a$ is the condition of applicability of perturbation theory in our example. For very large ε such that $|\varepsilon| \sim a$, we should expect that Eq. (8) becomes a bad approximation to the exact solution. One says that perturbation theory **breaks down** at $|\varepsilon| \sim a$.

How can we estimate the error of the approximation (8)? We can argue as follows. Perturbation theory works only when ε is sufficiently small so that each term in the series is much smaller than the previous term. If we use only the first two terms, $x_* \approx 1 + \varepsilon/a$, we are neglecting the third term, which is a correction of order ε^2/a^2. The third term, if included, will certainly improve the precision. Therefore, the error of the approximate solution $1 + \varepsilon/a$ can be estimated as $\Delta x \approx C\varepsilon^2/a^2$, where C is a constant roughly equal to $|a-3|/2$. Similarly, the error of the approximation (8) is approximately equal to the fourth term, which is of order $|\varepsilon/a|^3$.

2.3 Higher orders

It is clear that by using the method shown above, we can calculate all the coefficients in the expansion (5), one by one. Let us describe the procedure in more general terms. We assume the expansion

$$x_*(\varepsilon) = 1 + \sum_{n=1}^{N} A_n\varepsilon^n + O(\varepsilon^{N+1}),$$

which contains N unknown constants $A_1, ..., A_N$. This is called a **perturbative expansion** since it describes a small perturbation of the unperturbed solution $x = 1$. We then substitute this expansion into Eq. (4), expand everything in ε, disregarding terms of order ε^{N+1} or higher, and obtain an equation of the form

$$1 - 1 + (nA_1 - 1)\varepsilon + \left(aA_2 + \frac{a(a-1)}{2}A_1^2 - A_1\right)\varepsilon^2 + ... + (...)\varepsilon^N + O(\varepsilon^{N+1}) = 0. \tag{9}$$

Since ε is arbitrary, this equation can be satisfied only if *each* of the N coefficients at ε, ε^2, ..., ε^N is equal to zero.

To establish this more rigorously, we divide Eq. (9) by ε and take the limit as $\varepsilon \to 0$; we find $nA_1 - 1 = 0$. Thus the first nonvanishing term in Eq. (9) is of order ε^2. Then we can

divide by ε^2 and take the limit $\varepsilon \to 0$, which yields Eq. (7), etc. Thus, Eq. (9) means that the coefficient at each order in ε is equal to zero, independently of all other orders.

In this way, a single condition (9) yields a system of N equations for the N unknown constants. This system of equations is easy to solve because the first equation contains only A_1, the second equation contains only A_1 and A_2, and so on. Of course, the calculations will become more cumbersome for higher orders, since we will have to retain many more terms, but there is no difficulty in principle. These calculations are so straightforward that one can program a computer to perform these calculations symbolically. (A calculation is called **symbolic** if the computer is programmed to print the resulting formula containing variables such as ε and a, as opposed to a **numerical** calculation where the computer prints only a numerical value of x_*.)

2.4 Convergence and asymptotic expansions

In principle, one could calculate *all* the coefficients A_n, $n = 1, 2, 3, ...$, by the method shown above. Then one may ask whether the resulting *infinite* series

$$x_*(\varepsilon) = 1 + \sum_{n=1}^{\infty} A_n \varepsilon^n \tag{10}$$

will converge to the exact solution x_*. Unfortunately, the question of convergence is much more difficult to answer; to answer it, one needs some information about the *exact solution* of the perturbed equation. In some cases (such as Eq. (4) above, the series converges. However, in many cases, especially when applying perturbation theory to differential equations, the perturbative series does not converge. Nevertheless, a **truncated series** (i.e. the series where we take only a fixed, *finite* number of first terms of the perturbative expansion) always provides excellent precision when ε is very small, and hence can be successfully used in practice to determine approximate solutions. For instance, Eq. (8) is a very accurate approximation to the solution of Eq. (4) when $|\varepsilon/a| \ll 1$.

This is a somewhat paradoxical situation: A truncated series is useful because it gives a good approximation to $x_*(\varepsilon)$ at small ε, but the full infinite series does not actually converge to the exact value of $x_*(\varepsilon)$.

Let us consider this situation in some more detail. The sum of first N terms can be thought of as a function of ε and N,

$$\tilde{x}_*(\varepsilon, N) \equiv 1 + \sum_{n=1}^{N} A_n \varepsilon^n.$$

This function may have very different behavior as $N \to \infty$ at fixed ε, and as $\varepsilon \to 0$ at fixed N. In particular, the function $\tilde{x}_*(\varepsilon, N)$ provides a very precise approximation to the exact solution $x_*(\varepsilon)$ at fixed N and small ε. The error of this approximation is of order ε^{N+1}. What does it mean, in mathematical terms, that the error is "of order" ε^{N+1}? It means the following property,

$$\lim_{\varepsilon \to 0} \frac{\tilde{x}_*(\varepsilon, N) - x_*(\varepsilon)}{\varepsilon^N} = 0. \tag{11}$$

If this property holds, one says that the series $\tilde{x}_*(\varepsilon, N)$ is an **asymptotic expansion** of the solution $x_*(\varepsilon)$. Obviously, such a series is very useful in practice, since the precision of a truncated series is very good at sufficiently small ε. Nevertheless, it often happens that the infinite series diverges, $\lim_{N\to\infty} \tilde{x}_*(\varepsilon, N) = \infty$. There is no contradiction since these limits are taken in very different ranges of the variables ε, N. An asymptotic expansion may be either convergent or divergent.

The conclusion is that perturbation theory will give an *asymptotic* expansion, but not always a *convergent* expansion. However, in practice it is almost always impossible to obtain all the coefficients A_n (i.e. to derive a formula for A_n as a function of n). Moreover, one usually computes only two or three terms of the series, that is, one computes $\tilde{x}_*(\varepsilon, N)$ with $N = 2$ or, at most, $N = 3$. So the possible divergence at $N \to \infty$ is not important. It is much more important to obtain an estimate of the error and to determine the admissible values of ε for which the approximate solution $\tilde{x}_*(\varepsilon, N)$ is still precise at a fixed, small value of N.

2.5 How to guess the perturbative ansatz

The starting point of perturbation theory is a perturbative expansion such as Eq. (5). In other words, we are first *guessing* the solution to the problem in a form of a certain formula, or **ansatz**, containing ε in a certain way. We do not know in advance that this ansatz is correct; the justification comes later when we actually arrive at the solution (8). But it might happen that we have not guessed the ansatz correctly; then the procedure will not work.

For example, consider the equation

$$x^2 - 2(1+\varepsilon)x + 1 = 0. \tag{12}$$

This equation looks like a perturbation of the equation $x^2 - 2x + 1 = 0$, which has the solution $x = 1$. Let us assume the ansatz

$$x_*(\varepsilon) = 1 + A\varepsilon + O(\varepsilon^2) \tag{13}$$

and substitute it into Eq. (12), keeping only terms linear in ε. We find

$$-2\varepsilon = O(\varepsilon^2).$$

This condition cannot be satisfied because nothing can cancel the linear term 2ε. Therefore, the ansatz (13) is incorrect.

The reason for this trouble will become clear if we look at the exact solution of Eq. (12),

$$x_*(\varepsilon) = 1 + \varepsilon \pm \sqrt{2\varepsilon + \varepsilon^2}. \tag{14}$$

The leading correction is actually of order $\sqrt{\varepsilon}$ rather than of order ε, and the expansion of $x_*(\varepsilon)$ is of the form

$$x_* = 1 + A_1\sqrt{\varepsilon} + A_2\varepsilon + A_3\varepsilon^{3/2} + ...$$

Therefore, the correct perturbative ansatz must be of this form. After substituting such an ansatz into Eq. (12) and equating terms of equal order in ε, one gets $A_1 = \pm\sqrt{2}$, $A_2 = 1$, etc.

Now that we have the exact solution (14), it is easy to see that the perturbative expansion is the Taylor series for the analytic function $x_*(\sqrt{\varepsilon})$. Therefore, the perturbative expansion actually converges for small enough ε. But it would be difficult to judge the convergence of the expansion if we had only computed the first few coefficients $A_1, A_2, ..., A_n$ of the expansion and did not have a formula for the exact solution.

The conclusion is that one will run into trouble if one tries to use an incorrect perturbative expansion. When this happens, another expansion ansatz must be chosen, e.g. containing $\sqrt{\varepsilon}$ or $\ln \varepsilon$ instead of ε. One can often use physical intuition or qualitative considerations about the exact solution to guess the correct ansatz.

3 Perturbation theory for differential equations

Let us apply the method of perturbation theory to a differential equation. For example, consider the equation

$$\dot{x} + 2.33x = 0.000001x^3, \quad x(0) = 15. \tag{15}$$

It is clear that this equation can be thought of as a small perturbation of

$$\dot{x} + 2.33x = 0, \quad x(0) = 15,$$

whose exact solution can be easily found,

$$x(t) = 15e^{-2.33t}. \tag{16}$$

Therefore, we rewrite Eq. (15) as

$$\dot{x} + ax = \varepsilon x^3, \quad x(0) = A, \tag{17}$$

where we introduced a small parameter $\varepsilon \equiv 10^{-6}$ and (non-small) parameters $a \equiv 2.33$, $A \equiv 15$. Let us now apply the method of perturbation theory to this problem.

As the next step, we assume that the solution of Eq. (17) is a small perturbation of the solution (16), i.e.,

$$x(t) = Ae^{-at} + \varepsilon f_1(t) + O(\varepsilon^2), \tag{18}$$

where the function $f_1(t)$ needs to be determined. The initial condition for $f_1(t)$ is found by setting $t = 0$ in Eq. (18), which yields

$$A + \varepsilon f_1(0) + O(\varepsilon^2) = A.$$

Since the term linear in ε must vanish, we obtain $f_1(0) = 0$. We now substitute the ansatz (18) into Eq. (17) and expand to first order in ε:

$$\varepsilon \dot{f}_1 + \varepsilon a f_1 = \varepsilon A^3 e^{-3at} + O(\varepsilon^2).$$

All the terms linear in ε must cancel separately from terms quadratic in ε. Therefore, we obtain the equation

$$\dot{f}_1 + af_1 = A^3 e^{-3at}, \quad f_1(0) = 0.$$

The solution is

$$f_1(t) = A^3 \frac{e^{-at} - e^{-3at}}{2a}.$$

Therefore, the approximate solution of Eq. (17) is

$$x(t) \approx Ae^{-at} + \varepsilon A^3 \frac{e^{-at} - e^{-3at}}{2a}.$$

Let us now find a second-order correction. We start with the ansatz

$$x(t) = Ae^{-at} + \varepsilon f_1(t) + \varepsilon^2 f_2(t) + O(\varepsilon^3), \tag{19}$$

where $f_1(t)$ is already known and $f_2(t)$ is to be determined. The initial condition for $f_2(t)$ is $f_2(0) = 0$. Substituting this into Eq. (17), we find

$$\dot{f}_2 + af_2 = 3A^2 e^{-2at} f_1(t) = \frac{3A^5}{2a}\left(e^{-3at} - e^{-6at}\right), \quad f_2(0) = 0.$$

The solution is

$$f_2(t) = A^5 \frac{9e^{-at} - 15e^{-3at} + 6e^{-6at}}{20a^2}.$$

Therefore, the second-order approximate solution is

$$x(t) = Ae^{-at} + \varepsilon A^3 \frac{e^{-at} - e^{-3at}}{2a} + \varepsilon^2 A^5 \frac{9e^{-at} - 15e^{-3at} + 6e^{-6at}}{20a^2}. \tag{20}$$

3.1 Precision and limits of applicability

As a rule, the perturbative expansion provides good precision if its successive terms rapidly diminish in magnitude. Therefore, the condition of applicability of perturbation theory to Eq. (17) is

$$\left| \varepsilon A^3 \frac{e^{-at} - e^{-3at}}{2a} \right| \ll Ae^{-at}, \quad \text{for all } t.$$

It straightforwardly follows that $|\varepsilon| \ll 2aA^{-2}$. The condition for the quadratic term in Eq. (20) is

$$\varepsilon^2 A^5 \frac{\left|9e^{-at} - 15e^{-3at} + 6e^{-6at}\right|}{20a^2} \ll \left| \varepsilon A^3 \frac{e^{-at} - e^{-3at}}{2a} \right|,$$

which gives $|\varepsilon| \ll 9aA^{-2}/10$. It follows that perturbation theory is well within its limit of applicability for the values $\varepsilon = 10^{-6}$, $A = 15$, $a = 2.33$.

The precision of the approximate solution (20) can be estimated as follows. The first term is $x_0(t) = Ae^{-at}$, the second term is of order

$$\varepsilon A^3 a^{-1} e^{-at} \sim \frac{\varepsilon A^2}{a} x_0 \approx 10^{-4} x_0,$$

and the third term is of order

$$\varepsilon^2 A^5 a^{-2} e^{-at} \sim \left(\frac{\varepsilon A^2}{a}\right)^2 x_0 \approx 10^{-8} x_0.$$

If we computed the next term of the perturbative series, it would be of order $10^{-12} x_0$, but we did not include that term in Eq. (20). Therefore, the expected error is of order $10^{-12} x_0$.

3.2 Convergence

In principle, we can compute *all* the unknown functions $f_1(t)$, $f_2(t)$, ..., in the perturbative expansion

$$x(t) = e^{-at} + \sum_{n=1}^{\infty} \varepsilon^n f_n(t). \tag{21}$$

To compute each $f_n(t)$, we would need to solve a simple linear equation of the form

$$\dot{f}_n + af_n = (\ldots),$$

where the omitted expression in the right-hand side will depend on the functions f_1, ..., f_{n-1}. Therefore, we can easily

determine $f_n(t)$ if the previous functions f_1, ..., f_{n-1} are already found. As before, the calculations become more cumbersome at higher orders, but there is no difficulty in principle. A computer can be programmed to compute the result symbolically to any order.

The expansion (21) is *asymptotic* as $\varepsilon \to 0$, i.e. the partial sum

$$x_{(n)}(t) = e^{-at} + \sum_{n=1}^{N} \varepsilon^n f_n(t)$$

satisfies the equation up to terms of order ε^{N+1} and is therefore a good approximation to the exact solution when ε is small. We can show that this expansion *converges*. Indeed, in this case we can actually find the exact solution of Eq. (17) by separating the variables,

$$\int_A^x \frac{dx}{\varepsilon x^3 - ax} = \int_0^t dt = t.$$

After some calculations, this gives the solution

$$x(t) = Ae^{-at}\left[1 - \frac{\varepsilon A^2}{a}\left(1 - e^{-2at}\right)\right]^{-\frac{1}{2}}.$$

This is an analytic function of ε near $\varepsilon = 0$, therefore a Taylor expansion in ε will converge for small enough ε (more precisely, for $|\varepsilon| < aA^{-2}$). The expansion (21) is equivalent to a Taylor expansion of the above function in ε. Hence, we conclude that the perturbative expansion will converge for small enough ε.

3.3 Perturbative expansions with $\varepsilon = 1$

In the example considered in this section, we saw that perturbation theory works well as long as $|\varepsilon| A^2 \ll a$. If this condition holds, the perturbative expansion is useful even if the parameter ε is itself *not small*. Even if $\varepsilon = 10$, the perturbative expansion will be precise for sufficiently small A. For instance, we may set $a = \varepsilon = 1$ and conclude that the equation

$$\dot{x} + x = x^3 \tag{22}$$

may be solved using perturbation theory if the initial value $x(0) = A$ is small, $|A| \ll 1$.

This may appear paradoxical at first sight: we constructed the perturbative expansion under the assumption that ε is small, disregarded higher powers of ε, and now we want to use the result with $\varepsilon = 1$. The reason for the success of the perturbative expansion with $\varepsilon = 1$ can be seen by examining the formula (20). In that formula, every power of ε is multiplied also by A^2/a. In other words, the perturbative expansion is a series not in powers of ε but in powers of $\varepsilon A^2/a$. The value of ε does not actually need to be small.

The lesson is that one may apply perturbation theory successfully even to an equation such as Eq. (22), where no small parameters are apparently present. To apply perturbation theory, we may *introduce* a parameter $\varepsilon = 1$ and rewrite the equation as

$$\dot{x} + x = \varepsilon x^3, \quad x(0) = A. \tag{23}$$

The perturbative ansatz can still be formulated as in Eq. (19), as if ε is small, even though in reality $\varepsilon = 1$. Corrections of order ε^2 in the final result (20) *will be* actually smaller than terms of order ε, not because ε itself is small, but because every ε is multiplied by a small number A^2. One says that the parameter ε is **formal**; by this one means that ε serves merely as a bookkeeping parameter that measures the smallness of corrections. Of course, it takes a certain amount of intuition to realize that the parameter ε should be attached to the x^3 term rather than to some other term in Eq. (22). This intuition usually comes from physical considerations; in the case of Eq. (22), one expects that the x^3 term is insignificant when the initial value $x(0)$ is very small, especially since the solution of $\dot{x} + x = 0$ exponentially decays with time.

However, I would like to stress that perturbation theory does involve a certain amount of guessing, because one needs to start from a certain perturbative ansatz, and there are no absolute rules about choosing that ansatz. How does one know that the ansatz is guessed correctly? Proving the convergence of a perturbative expansion is difficult unless one actually knows something about the exact solution. In the absence of a proof of convergence, one can justify the perturbative expansion by verifying that higher-order terms of the expansion are smaller in magnitude than lower-order terms. This will show the limits of applicability of perturbation theory and also provide an estimate of the resulting precision.

4 Anharmonic oscillations

In the preceding sections, we have considered "toy" perturbation problems where the exact solution is in fact known and there is no need to use perturbation theory. The analysis of these examples helps understand possible pitfalls of guessing the perturbative expansion in various cases. We are now ready to consider the problem of anharmonic oscillations, where exact solutions are not known.

As a first example of an anharmonic oscillator, consider a point mass m moving in the potential

$$V(x) = \frac{1}{2}kx^2 + \frac{1}{4}m\varepsilon x^4,$$

where ε is a small parameter, while k is not small. This potential has a minimum at $x = 0$ (for all ε), so we expect oscillatory behavior near this stable equilibrium point. Of course, oscillations will not proceed according to a sine curve such as $x(t) = \sin \omega_0 t$. One says that in this case the oscillations are **anharmonic** (i.e. not harmonic).

It is clear that the potential $V(x)$ is a small perturbation of the harmonic potential $V_0(x) = \frac{1}{2}kx^2$. Suppose that the oscillator was released from rest at a position $x = A \neq 0$. Then the equation of motion is

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{24}$$

where $\omega_0^2 \equiv k/m$ is the frequency of oscillations in the unperturbed potential $V_0(x)$.

4.1 Failure of a simple perturbation ansatz

Let us now apply perturbation theory to Eq. (24). The unperturbed solution is

$$x_0(t) = A\cos\omega_0 t.$$

As a first attempt, let us assume a perturbative expansion of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ..., \tag{25}$$

where $x_1(t)$, $x_2(t)$, ... are unknown functions. Substituting this ansatz into Eq. (24), we find

$$\ddot{x}_1 + \omega_0^2 x_1 = -x_0^3,$$
$$\ddot{x}_2 + \omega_0^2 x_2 = -3x_0^2 x_1.$$

The initial conditions for the functions $x_1(t)$ and $x_2(t)$ are $x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$ since the initial condition $x(0) = A$, $\dot{x}(0) = 0$ is already exactly satisfied by the unperturbed solution $x_0(t)$. Note that

$$x_0^3 = A^3 \cos^3 \omega_0 t = A^3 \frac{\cos 3\omega_0 t - 3\cos \omega_0 t}{4},$$

therefore we obtain

$$x_1(t) = \frac{A^3}{32\omega_0^2} \left[\cos 3\omega_0 t - \cos \omega_0 t - 12\omega_0 t \sin \omega_0 t\right].$$

Thus the approximate first-order solution is

$$x(t) \approx A\cos \omega_0 t + \varepsilon \frac{A^3}{32\omega_0^2} \left[\cos 3\omega_0 t - \cos \omega_0 t - 12\omega_0 t \sin \omega_0 t\right].$$

We omit the calculation of $x_2(t)$ because there is already a problem with the above expression for $x_1(t)$. Namely, it includes the term $t \sin \omega_0 t$ which describes oscillations with a growing amplitude. Clearly, the term $\varepsilon t \sin \omega_0 t$ will become arbitrarily large at late times, thus breaking down the perturbation theory, even if ε is very small. Such terms that grow at late times are called **secular** terms (from the Latin *saeculum* = century; the mental image is, perhaps, that a small contribution that persists over centuries will eventually grow into a large contribution). The reason for the appearance of the secular term is, technically, the presence of $\cos \omega_0 t$ in the right-hand side of the equation for $x_1(t)$, which is analogous to the presence of a periodic force whose frequency resonates with the frequency of the oscillator. The resonance leads to a linear growth of the oscillation amplitude.

The problem of secular terms is generic to any situation with anharmonic oscillations. Secular terms may not appear in all orders in ε; for instance, if we apply perturbation theory to the equation

$$\ddot{x} + \omega_0^2 x = \varepsilon x^2,$$

secular terms will first appear only at the second order in ε. This, of course, merely postpones the problem.

Secular terms appearing in a perturbative expansion are unsatisfactory for two reasons. First, they limit the applicability of perturbation theory to small values of time t. Perturbation theory breaks down at $|\varepsilon t| \sim \omega_0 A^{-2}$. Higher-order coefficients in the perturbative expansion will also contain secular terms; thus, we cannot improve the precision at late times, no matter how many orders of ε we compute. Second, an approximate solution containing secular terms does not reproduce the correct behavior of anharmonic oscillations: Since the total energy of the system is conserved, the amplitude of oscillations cannot actually increase without bound. We conclude that the perturbative ansatz (25) is incorrect. Let us now come up with an improved ansatz.

4.2 Lindstedt-Poincaré method

We shall begin with some qualitative considerations. When the potential is unperturbed ($\varepsilon = 0$), the oscillations are harmonic and their frequency ω_0 is independent of the amplitude A. In the presence of the perturbation (i.e. with $\varepsilon \neq 0$), we would expect that the frequency of oscillations *will depend* on the amplitude A as well as on ε.

Since we expect that the perturbed solution is still oscillatory but with a slighly different frequency, it makes sense to incorporate a change of frequency in the perturbative ansatz:

$$x(t) = A\cos \omega t + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ..., \tag{26}$$

where

$$\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + ... \tag{27}$$

is the "perturbed" frequency of the oscillations, and the constants ω_1, ω_2, ... are to be determined together with the unknown functions $x_1(t)$, $x_2(t)$, ...

The perturbation method we shall use is known as the **Lindstedt-Poincaré method**. This method uses the expansions (26)-(27) with a further mathematical trick: one changes the time variable from t to the dimensionless "phase" variable $\phi \equiv \omega t$, where ω is the *perturbed* frequency. After this change of variable, the perturbative ansatz becomes

$$x(\phi) = A\cos \phi + \varepsilon x_1(\phi) + \varepsilon^2 x_2(\phi) + ..., \tag{28}$$

while Eq. (24) needs to be rewritten through derivatives $dx/d\phi$ and $d^2x/d\phi^2$. Let us denote these derivatives by a prime, then we have

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{d\phi^2}\omega^2 \equiv \omega^2 x'',$$

and finally Eq. (24) becomes

$$\left(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + ...\right)^2 x'' + \omega_0^2 x + \varepsilon x^3 = 0. \tag{29}$$

The initial conditions for this equation are $x(0) = A$ and $x'(0) = 0$.

Now we can apply the perturbation ansatz (28) straightforwardly to Eq. (29). Keeping only the linear terms in ε, we obtain the following equation for $x_1(\phi)$:

$$x_1'' + x_1 = 2A\frac{\omega_1}{\omega_0}\cos\phi - \frac{A^3}{\omega_0^2}\cos^3\phi, \quad x_1(0) = x_1'(0) = 0. \tag{30}$$

The solution of this equation will have secular terms if the right-hand side contains $\cos\phi$ or $\sin\phi$. Since

$$\cos^3\phi = \frac{\cos 3\phi - 3\cos\phi}{4},$$

the right-hand side of Eq. (30) is

$$\left(2A\frac{\omega_1}{\omega_0} + \frac{3}{4}\frac{A^3}{\omega_0^2}\right)\cos\phi - \frac{A^3}{4\omega_0^2}\cos 3\phi.$$

A secular term will appear if the coefficient at $\cos\phi$ is nonzero. However, the value of ω_1 is still undetermined; so we can *choose* the value of ω_1 to avoid the appearance of secular terms. This is the key point of the Lindstedt-Poincaré method. It is clear that we need to choose

$$\omega_1 = -\frac{3}{8}\frac{A^2}{\omega_0}.$$

The solution of Eq. (30) is then

$$x_1(\phi) = \frac{A^3}{32\omega_0^2}\left(\cos 3\phi - \cos\phi\right),$$

and thus the approximate first-order solution of the original equation is

$$x(t) \approx A\cos\omega t + \frac{\varepsilon A^3}{32\omega_0^2}\left(\cos 3\omega t - \cos\omega t\right), \qquad (31)$$
$$\omega \approx \omega_0 - \frac{3}{8}\frac{\varepsilon A^2}{\omega_0}.$$

The solution (31) is well-behaved for all t and describes oscillations with a slightly perturbed frequency and with slight deviations from pure cosine shape.

4.3 Precision of approximation

Of course, the solution (31) is still *approximate*. The major improvement over the initial ansatz (25) is that there are no secular terms, i.e. the amplitude of oscillations does not grow with time. However, the next correction to the frequency is of order ε^2, so we expect that the first-order solution will deviate significantly from the exact solution after times $t \sim \varepsilon^{-2}$. Therefore, the solution (31) can be expected to be precise only for $t \ll \varepsilon^{-2}$. Moreover, the standard condition of applicability of perturbation theory is that the higher-order terms are smaller, which gives

$$\frac{|\varepsilon|\, A^3}{32\omega_0^2} \ll A, \quad \frac{3}{8}\frac{|\varepsilon|\, A^2}{\omega_0} \ll \omega_0.$$

This condition is satisfied if $|\varepsilon|\, A^2 \ll \omega_0^2$, which means that either the perturbation in the potential is small, or the amplitude A of oscillations is small. Physically, this condition means that the change in the energy due to the perturbation of the potential, $\frac{1}{4}\varepsilon A^4$, is much smaller than the typical energy of the unperturbed oscillator, $\frac{1}{2}\omega_0^2 A^2$. Note also that negative values of ε such that $\varepsilon < -\omega_0^2 A^{-2}$ are physically significant. For these values, the solution $x(t)$ has the initial acceleration

$$\ddot{x}(0) = -\omega_0^2 A + |\varepsilon|\, A^3 > 0,$$

which means that the oscillator does not move towards the stable equilibrium point $x = 0$ from its the initial position, $x(0) = A$. In this case the motion cannot be pictured as merely a small perturbation of the harmonic oscillation. These arguments show that we should not expect perturbation theory to yield any good results if $\left|\varepsilon A^2 \omega_0^{-2}\right| \geq 1$.

The precision of the approximation (31) can be estimated as follows. The first correction is of order $|\varepsilon|\, A^2\omega_0^{-2}$ times the unperturbed solution. Therefore, one expects that the next correction will be of order $|\varepsilon|^2 A^4 \omega_0^{-4}$ times the unperturbed solution. So the relative error of the approximation (31) is of order $|\varepsilon|^2 A^4 \omega_0^{-4}$.

In reality, the error of the approximation will be some constant, say C_1, times $|\varepsilon|^2 A^4 \omega_0^{-4}$. It is not easy to compute C_1 rigorously, but our estimates show that C_1 might be a number of order 1.

Using the Lindstedt-Poincaré method, one can find further terms in the perturbative expansion (28)-(29). At the n-th step, the function $x_n(t)$ will be determined from an equation such as

$$x_n'' + x_n = (\ldots),$$

where the terms in the right-hand side will depend on previously found constants ω_1, ..., ω_{n-1} and functions $x_1(t)$, ..., $x_{n-1}(t)$. The value of ω_n will be determined from the condition that no secular terms should arise in this equation. The n-th order solution will be valid until times $t \sim \varepsilon^{-n-1}$. For times $t \ll \varepsilon^{-n-1}$, the error of the n-th order solution is $C_n \left|\varepsilon A^2 \omega_0^{-2}\right|^{n+1}$ times the unperturbed solution, where C_n is a proportionality constant that may be difficult to compute.

Note that the perturbation series may or may not converge (for fixed ε) depending on how quickly the constant C_n grows with n. If C_n grows too quickly for large n, say as $n!$, we will obtain a divergent series. Nevertheless, a fixed number of terms of this series will be a useful approximation to the solution.

4.4 Discussion

Let us summarize the results we obtained. We have applied perturbation theory to Eq. (24) with a straightforward ansatz (25), but the attempt failed because of the appearance of secular terms such as $t\cos\omega_0 t$. The second attempt was to use an improved ansatz (26)-(27), which takes into account a change in the oscillation frequency,

$$\omega_0 \to \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots,$$

due to the perturbation of the potential. However, perturbation theory based on Eq. (24) and the ansatz (26)-(27) would have failed, had we not used the Lindstedt-Poincaré method, which mandates a change of variable, $t \to \phi = \omega t$, involving the perturbed frequency ω. Equation (24) is rewritten in the new variables $x(\phi)$, yielding Eq. (29), and straightforward perturbation theory can now be applied. At each order in ε, one can choose the constants ω_1, ω_2, ..., so that no secular terms appear. The result is a well-behaved approximate solution for $x(t)$. The n-th order solution is valid until times $t \sim \varepsilon^{-n-1}$, as long as the amplitude of oscillations A and the parameter ε are sufficiently small so that $|\varepsilon|\, A^2 \ll \omega_0$.

What is the reason for the failure of perturbation theory without the Lindstedt-Poincaré method? We can easily understand this by looking at the solution (31). The "straightforward" perturbative ansatz (25) is a Taylor expansion of that solution in ε. For instance, the term $A\cos\omega t$ will be expanded as

$$A\cos\left[(\omega_0 + \varepsilon\omega_1)\, t\right] = A\cos\omega_0 t - A\varepsilon\omega_1 t\sin\omega_0 t + O(\varepsilon^2).$$

This expansion is only meaningful for small enough ε such that $|\varepsilon|\,\omega_1 t \ll 1$: The second term grows without bound at late times, so the expansion breaks down at $t \sim \varepsilon^{-1}$. Now it becomes clear why the secular term $t\sin\omega_0 t$ appears when one uses a simple perturbative expansion in ε.

The Lindstedt-Poincaré method can be also applied to any equation of the form

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}),$$

where ε is the expansion parameter and $f(x, \dot{x})$ is some expression involving x and $\dot{x}$, or to systems of such equations.

5 Suggested literature

E. J. Hinch, *Perturbation methods* (Cambridge University Press, 1995). This is a short introductory book with many examples.

A. H. Nayfeh, *Perturbation methods* (Wiley, 1973). This is a more advanced book showing a great multitude of tricks one needs to apply in order to develop a successful perturbation theory in different cases.

5.1 Remarks on Landau and Lifshitz §28

I *warn against* reading Landau and Lifshitz's *Mechanics* (third edition, Butterworth - Heinemann, 1976, which is the last English edition) if it is your first source of information on this topic. They discuss anharmonic oscillations in §28 without using the Lindstedt-Poincaré method and obtain correct results through a slightly incorrect mathematical reasoning.

Since Landau and Lifshitz is an extremely well respected textbook that has very few errors, I give full details of the calculation. The problem is that their Eq. (28.12) cannot be derived from the preceding Eq. (28.11) merely, as they say, by "omitting terms of above the second order of smallness": one needs to suppress certain terms without explanation.

Their Eq. (28.11) is fully equivalent to Eq. (28.9), which we may rewrite (explicitly introducing the parameter $\varepsilon = 1$) as

$$\ddot{x} + \omega_0^2 x = -\varepsilon \left(\alpha x^2 + \beta x^3\right). \tag{32}$$

They use the ansatz

$$x(t) = a\cos\omega t + x^{(2)}(t), \quad \omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + ...,$$

where $x^{(2)}$, $\omega^{(1)}$, etc., are quantities of higher order. Let us make this explicit by rewriting the ansatz as

$$x(t) = a\cos\omega t + \varepsilon x_1(t), \qquad \omega = \omega_0 + \varepsilon\omega_1 + ... \tag{33}$$

Now if we follow Landau and Lifshitz and substitute the ansatz (33) into Eq. (32), we will find

$$-2a\varepsilon\omega_0\omega_1\cos\omega t + \varepsilon\left(\ddot{x}_1 + \omega_0^2 x_1\right) = -\varepsilon\alpha a^2\cos^2\omega t + O(\varepsilon^2).$$

We equate the first-order terms and obtain an equation for $x_1(t)$,

$$\ddot{x}_1 + \omega_0^2 x_1 = 2a\omega_0\omega_1\cos\omega t - \alpha a^2\cos^2\omega t. \tag{34}$$

At this point, we observe that we have not yet fully expanded the right-hand side of Eq. (34) in ε because the "ω" in $\cos\omega t$ still contains the term $\varepsilon\omega_1$. The crucial point is that the correct result for $x_1(t)$, which is their Eq. (28.12),

$$x_1(t) = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega t,$$

will be obtained only if we expand ω in the first term in the right-hand side of Eq. (34) but *do not expand* ω in the second term (even though we expanded it just a second ago)!

Indeed, our Eq. (34) is identical to the unnumbered equation in Landau and Lifshitz just after Eq. (28.11). If we *do* expand the $\cos\omega t$ everywhere (as we should, since we did expand $\alpha x^2 + \beta x^3$ in ε), we will obtain

$$\cos\omega t = \cos\omega_0 t - \varepsilon\omega_1 t\sin\omega_0 t + ...$$

and then Eq. (34) becomes

$$\ddot{x}_1 + \omega_0^2 x_1 = 2a\omega_0\omega_1\cos\omega_0 t - \alpha a^2\cos^2\omega_0 t + O(\varepsilon).$$

The terms $O(\varepsilon)$ need to be neglected; in other words, we merely replace ω by ω_0 in the right-hand side of Eq. (34). Then we obtain the no-resonance condition, $\omega_1 = 0$, and the equation for $x_1(t)$ whose solution is

$$x_1(t) = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega_0 t.$$

In other words, we will not obtain Eq. (28.12) of Landau and Lifshitz because our $x_1(t)$ will contain $\cos 2\omega_0 t$ while theirs contains $\cos 2\omega t$.

In order to obtain $\cos 2\omega t$ in the expression for $x_1(t)$, we *must not* replace the term $\cos^2\omega t$ by $\cos^2\omega_0 t$ in the right-hand side of Eq. (34). However, in order to derive the no-resonance condition, $\omega_1 = 0$, we must replace the term $\cos\omega t$ by $\cos\omega_0 t$ in the same equation. If we do not replace $\cos\omega t$ by $\cos\omega_0 t$, we will not get any resonance since the left-hand side of Eq. (34) contains the unmodified frequency ω_0, while the right-hand side contains a different frequency, ω. Then our solution $x_1(t)$ would have been

$$x_1(t) = -\frac{a}{\varepsilon}\cos\omega t + a\frac{\omega_1}{\omega_0}\cos\omega t - \frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{6\omega_0^2}\cos 2\omega t + O(\varepsilon),$$

which is of course also incorrect because of the presence of the first two terms. We have no grounds to demand that these terms be absent since they are not secular terms (not a resonance). Note especially the peculiar term of order $1/\varepsilon$: No choice of ω_1 will make this term disappear.

The only way to derive Eq. (28.12) from Eq. (28.11) in Landau and Lifshitz is to "selectively" replace ω by ω_0 in some terms but not in other terms. This is mathematically incorrect, since we are not allowed to throw away some terms of order ε while keeping other terms of the same order. Since we have performed a mathematically incorrect calculation, we cannot be sure that the result is correct. In contrast, the Lindstedt-Poincaré method is conceptually simpler and performs the derivation without any mathematically incorrect operations.