Merkzettel zu Nabla, Grad, Div und Rot

a) Definitionen

$$\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \text{Nabla-Operator}$$

$$grad\ U(\vec{r}) := \nabla \cdot U(\vec{r}) \equiv \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \qquad \text{Gradient}$$

$$div\ \vec{v}(\vec{r}) := \nabla \cdot \vec{v}(\vec{r}) \equiv \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad \text{Divergenz}$$

$$rot\ \vec{v}(\vec{r}) := \nabla \times \vec{v}(\vec{r})$$

$$\equiv \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \qquad \text{Rotation}$$

b) Produktregeln

$$grad\ (U \cdot V) = U\ grad\ V + V\ grad\ U \qquad div\ (U \cdot \vec{v}) = U\ div\ \vec{v} + \vec{v}\ grad\ U$$

$$rot\ (U \cdot \vec{v}) = U\ rot\ \vec{v} - \vec{v} \times grad\ U \qquad div\ (\vec{u} \times \vec{v}) = \vec{v} \cdot rot\ \vec{u} - \vec{u} \cdot rot\ \vec{v}$$

c) implizite Ableitungen

$$grad\ U(V(\vec{r})) = \frac{dU}{dV} \cdot grad\ V(\vec{r}) \qquad grad\ U(r) = \frac{dU}{dr} \cdot \frac{\vec{r}}{r}$$

$$div\ \vec{u}(V(\vec{r})) = \frac{d\vec{u}(\vec{r})}{dV} \cdot grad\ V(\vec{r}) \qquad div\ \vec{u}(r) = \frac{d\vec{u}}{dr} \cdot \frac{\vec{r}}{r}$$

$$rot\ \vec{u}(V(\vec{r})) = -\frac{d\vec{u}(\vec{r})}{dV} \times grad\ V(\vec{r}) \qquad rot\ \vec{u}(r) = -\frac{d\vec{u}}{dr} \times \frac{\vec{r}}{r}$$

d) zweite Ableitungen

$$rot\ grad\ U = (\nabla \times \nabla)\ U \equiv 0 \qquad div\ rot\ \vec{v} = \nabla \cdot (\nabla \times \vec{v}) \equiv 0$$

$$rot\ rot\ \vec{v} = grad\ div\ \vec{v} - \triangle \vec{v}$$

$$\triangle := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{Laplace-Operator}$$

e) Integralsätze

$$\int_V dV\ div\ \vec{v} = \oint_{(V)} d\vec{A} \cdot \vec{v} \qquad \text{Integralsatz von Gauss}$$

Integral von $div\ \vec{v}$ über Volumen V = Integral von $\vec{v}$ über die Oberfläche (V)

$$\int_A d\vec{A} \cdot rot\ \vec{v} = \oint_{(A)} d\vec{s} \cdot \vec{v} \qquad \text{Integralsatz von Stokes}$$

Integral von $rot\ \vec{v}$ über Fläche A = Integral von $\vec{v}$ entlang des Umfangs (A)