Lehrstuhl für Theoretische Nanophysik
Dipl.-Phys. P. Kroiss
Dr. V. Alba
Prof. Dr. L. Pollet

2nd Exercise Sheet Many-Body Physics

Will be discussed in the week of May 6-10.

Exercise 1: Green function of a free particle in 1d

Compute the retarded Green function for a free particle in 1d with dispersion $E(k) = k^2/(2m)$. Answ:

$$G^R(x_f, x_i, E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x_f - x_i)}}{E - k^2/(2m) + i0} \tag{1}$$

Calculate the integral over k for $E < 0$. Perform analytical continuation to positive E over the upper half plane.

The retarded Green function in the (k, E) domain can be written as (show)

$$G^R(k, E) = \frac{1}{E + i0 - k^2/(2m)} \tag{2}$$

It can be thought of as the matrix elements of the operator $\hat{G}^R(E)$,

$$\langle k| \frac{1}{E + i0 - \hat{H}} |k'\rangle = \frac{2\pi\delta(k - k')}{E + i0 - k^2/(2m)} \tag{3}$$

but where the delta-function is absent in the matrix elements.

Exercise 2: Scattering off a δ-potential in 1d: bound states

Consider

$$\hat{H} = \frac{-1}{2m}\frac{d^2}{dx^2} + \lambda\delta(x) = \hat{H}_0(x) + \hat{V}(x). \tag{4}$$

Introduce

$$\hat{G}_0^R = \frac{1}{E + i0 - \hat{H}_0} \tag{5}$$

and the T-matrix

$$\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \hat{V}\hat{G}_0\hat{V}\hat{G}_0\hat{V} + \dots \tag{6}$$

Evaluate

$$\Pi = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\langle k|G_0^R|k'\rangle = \int \frac{dk}{2\pi}\frac{1}{E + i0 - k^2/(2m)}, \tag{7}$$

for both positive and negative energies. Show that $\hat{G}^R = \hat{G}_0^R + \hat{G}_0^R\hat{T}\hat{G}_0^R$ and calculate $\langle k|\hat{T}|k'\rangle$. When is there a bound state?

Exercise 3: Scattering off a δ-potential in 3d

Consider now the problem of scattering in 3D off a delta function potential,

$$\hat{H} = \frac{-1}{2m}\frac{d^2}{\nabla^2} + g\delta(\boldsymbol{r}) = \hat{H}_0(x) + \hat{V}(x). \tag{8}$$

This will require evaluation of

$$\Pi = \int_\Lambda \frac{d^3k}{(2\pi)^3}\frac{1}{E - k^2/(2m) + i0}, \tag{9}$$

which is UV divergent. One therefore introduces a cutoff Λ for k. So, unlike the delta function in 1d, the delta function in 3d does not represent a well defined potential. We should hence think of a potential which is g at low momenta, but quickly decays to zero for momenta larger than Λ, with Λ the scale set by the inverse of the extent of the potential.

- Show that Π can be written as

$$\Pi = -\frac{m\Lambda}{\pi^2} - \frac{2mE}{2\pi}\sqrt{\frac{m}{-2E}}. \tag{10}$$

- Compute the scattering amplitude $f = -\frac{m}{2\pi}T$, where T is the T-matrix. Show that it is angle independent (fully s-wave symmetric). Obtain the scattering length a defined by $a = -f(p = 0)$.
- When are there bound states? What happens for $g = -\pi^2/(m\Lambda)$?