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7th Exercise Sheet Many-Body Physics

Will be discussed in the week of June 10-14.

Exercise 1: A single damped classical harmonic oscillator

Consider a single damped harmonic oscillator,

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F^{\text{ext}}(t) \tag{1}$$

with $F^{\text{ext}}(t) = F_0\cos(\Omega t)$. Define the retarded response by

$$x(t) = \int_{-\infty}^{+\infty} dt' \chi_{\text{ret}}(t-t')F^{\text{ext}}(t'). \tag{2}$$

Show that

- χ_{ret} can be considered a Green function for the left hand side of Eq. 1 subject to $\chi_{\text{ret}}(t-t') = 0$ for $t < t'$.
- Compute the real χ' and imaginary χ'' parts of $\chi(\omega)$, which is the Fourier transform of the retarded response.
- Express the real and imaginary parts as Fourier transforms of a linear combination of $\chi(t)$ and $\chi(-t)$. Which one is invariant under $t \to -t$?
- Compute the dissipation $\frac{dW}{dt} = F(t)\frac{dx}{dt}$ and evaluate it at zero frequency. What shpae does it have? Why does the χ' not enter?
- Establish the Kramers-Kronig relations between χ' and χ''

Exercise 2: Fluctuation-dissipation theorem

Consider a quantum system H at finite temperature T in equilibrium subject to an external perturbation $F^{\mathrm{ext}}(t)$ for $t > t_0$ coupling to a quantity A of the system via

$$\hat{H}^{\mathrm{pert}}(t) = \hat{A} F_0 e^{i\Omega t} + \hat{A}^\dagger F_0^* e^{-i\Omega t}. \tag{3}$$

Compute in linear response the change in energy, $\langle \frac{dH^{\mathrm{full}}(t)}{dt} \rangle$, with $H^{\mathrm{full}}(t) = H + H^{\mathrm{pert}}(\mathrm{t})$. Express your result in terms of the imaginary part of the (retarded) response.

Exercise 3: Kubo formula for electrical conductivity

When applying an electric field to a solid, a current will be created. An experimentalist would like to know the induced current J that he or she can thus measure. However, the applied electric field will also create an internal electric field and internal currents that have to be treated carefully. We will work at $T = 0$. The conductivity σ is definied by the response to the *total* electrical field in the system,

$$J_\alpha(\boldsymbol{r}, t) = \sum_\beta \sigma_{\alpha,\beta} E_\beta(\boldsymbol{r}, t) = \sum_\beta \sigma_{\alpha,\beta} E_\beta e^{i(\boldsymbol{q}\cdot\boldsymbol{r} - \omega t)}. \tag{4}$$

The latter equality is equivalent to making a coarse-grained assumption such that short-range physics is irrelevant and we are only interested in the long-wavelength response and small $|\boldsymbol{q}|$. α and β denote spatial directions (x, y, z), and the assumption of a single mode for the electric field is compatible with linear response.

- Write down the interaction Hamiltonian that couples the current operator j_α (proportional to the average momentum operator $\frac{1}{V}\sum_i \langle p_{i\alpha} \rangle$) to the electric field by expressing the latter in terms of the vector potential A. Assume transverse fields, drop terms in A^2 and work in the Coulomb gauge.
- The measured or induced current J_α is proportional to the average velocity of all electrons, times the charge e. Perform the minimal substitution on the momentum operator and find that the measured current consists of a part proportional to the electric field and a second part $J^{(2)}_\alpha(\boldsymbol{r}, t) = \langle j_\alpha(\boldsymbol{r}, t) \rangle$ that is the average of the current operator, which needs further manipulations.
- Evaluate $J^{(2)}_\alpha(\boldsymbol{r}, t)$ in linear response, and bring it in the form of Eq. 4.
- Finally, we need to coarse grain and average over $\boldsymbol{r}$ to eliminate atomic fluctuations, so perform $\frac{1}{V}\int d^3r \ldots$. You should end up with the Kubo

formula for the electrical conductivity,

$$\sigma_{\alpha,\beta}(\boldsymbol{q},\omega) = \frac{1}{\omega V}\int_0^\infty dt e^{i\omega t}\langle\psi|\left[j^\dagger_\alpha(\boldsymbol{q},t), j_\beta(\boldsymbol{q},t')\right]|\psi\rangle + i\frac{n_0 e^2}{m\omega}\delta_{\alpha\beta}, \quad (5)$$

where $|\psi\rangle$ is the ground state of the many-body Hamiltonian H, containing all interactions in the solid except the interaction with the vector potential. Note that the conductivity is an intrinsic property of the ground state of the system and has no mentioning of a photon field in it.

- When computing the dc ($\omega = 0$) conductivity, you have to take the limit $q \to 0$ first and only then $\omega \to 0$. Argue why the limits do not commute. Convince yourself that the dc conductivity does not diverge (and is real).