

The energetics of simple circular loops will be applied in numerous places throughout the remainder of the chapter and especially in our consideration of transcriptional regulation and DNA packaging.

As yet, our analysis has been for the case in which the entire beam of length L is uniformly deformed into a single arc of a circle with radius R . More generally, we will be interested in states of deformation in which the local curvature differs from one point to the next. In this case, we invoke a *locality* assumption in which we pretend that the energy of a given material particle is that of the beam described above and for which the curvature is constant. The total energy is gotten by adding up the energy material particle by material particle and is of the form

$$E_{bend} = \frac{K_{eff}}{2} \int_0^L ds \frac{1}{R(s)^2}, \quad (10.8)$$

where we have introduced the flexural rigidity $K_{eff} = EI$, which embodies both material parameters (that is, the Young modulus E) and the geometric shape (through the geometric moment I). This equation is the mathematical embodiment of the cartoon shown in Figure 10.4, where we show how each point on the beam can be thought of locally as part of a circle. There are several other convenient ways of writing eqn 10.8 that will arise in the remainder of the book. First, we note that $\kappa(s) = 1/R(s)$, where we have defined the curvature $\kappa(s)$. This implies that we can rewrite the equation as

$$E_{bend} = \frac{K_{eff}}{2} \int_0^L \left| \frac{dt}{ds} \right|^2 ds, \quad (10.9)$$

where we use the fact that the curvature can itself be written as the derivative of the tangent vector.

10.2.2 Beam Theory and the Persistence Length: Stiffness is Relative

Thermal Fluctuations Tend to Randomize the Orientation of Biological Polymers

One intriguing theme that we will return to repeatedly throughout the book (already introduced in Section 5.1.1 on p. 168) is the idea of a competition between thermal effects and deterministic forces. For example, in Figure 9.11 (p. 340) we introduced the Bjerrum length as the length scale over which charges can wander from a protein without offending the Coulomb interaction too significantly. Here we examine another example of this same kind of argument in which a biological polymer such as DNA is kicked around into various different orientations as a result of interactions with the surrounding fluid. In this case, it is the elastic forces rather than the Coulomb forces that set the length scale over which such fluctuations are tolerated. In particular, the intuition corresponding to this idea is that if the polymer is too short, it will be indifferent to thermal fluctuations. By way of contrast, for very long polymers, the orientation at one extremity is completely indifferent to that at the other.

The Persistence Length Is the Length Over Which a Polymer Is Roughly Rigid

The competition described above between thermal fluctuations and the energetic cost associated with beam bending is succinctly captured in the emergence of a single length scale, namely, the persistence length.

The persistence length is a measure of the competition between the entropic parts of the free energy which tend to randomize the orientation of the polymer and the energetic cost of bending. Throughout physical biology, there are a variety of different length scales which arise that, at the deepest level, reflect the interplay between energy and entropy. Generally, length scales that reflect the competition between thermal and deterministic energies can be estimated by equating the deterministic energy cost for the particular mechanism of interest to $k_B T$, the thermal energy. When elastic energies are competing with thermal fluctuations, the relevant comparison is

$$k_B T \approx \frac{EIL}{2R^2}, \quad (10.10)$$

where L is the length of the fragment of interest. Roughly speaking, the persistence length ξ_p is that length of polymer for which the radius of curvature is equal to the length of polymer itself. Hence, if we set L and R both equal to ξ_p , we see that our estimate for the persistence length is given by

$$\xi_p \approx \frac{EI}{2k_B T}. \quad (10.11)$$

The Persistence Length Characterizes the Correlations in the Tangent Vectors at Different Positions Along the Polymer

An alternative view of the persistence length is to think of biological polymers from the standpoint of the geometry of space curves. There are a number of different ways of characterizing the mathematics of space curves. One way is via correlation functions which measure the extent to which the geometry of one part of the polymer is correlated with some other part.

To be concrete, consider a space curve in parametric representation given by the vector $\mathbf{r}(s)$, where s is the arclength parameter. The function $\mathbf{r}(s)$ is a rule that assigns a position vector for every value of the arclength s . Such a configuration is shown in Figure 10.7. The question we pose concerning a given space curve is the nature of the tangent-tangent correlation function defined as $g(s) = \langle \mathbf{t}(\tau + s) \cdot \mathbf{t}(\tau) \rangle$. Our notation is built around the unit tangent vector, $\mathbf{t}(s)$, which is defined as the unit vector tangent to the curve at arclength s . The tangent-tangent correlation function as defined above is thus far a purely mathematical notion and provides a measure of the relation between the tangents at arclength τ and arclength $\tau + s$. When the correlation function is 1, the tangent vectors are parallel.

This idea was made explicit in Figure 8.6 (p. 292), where we showed the genome of a bacterium that had been released from the cell. In the inset to that figure, we labeled the persistence length with small white arrows, each with a length equal to that of the persistence length of DNA. These arrows reveal that on the scale of tens of nanometers, the tangents are correlated while on the scale of microns they are not.

The Persistence Length Is Obtained by Averaging Over All Configurations of the Polymer

Qualitatively, we see that the actual numerical value of the persistence length depends upon the stiffness of the filament. Microtubules and DNA do not have the same persistence length with DNA characterized by a persistence length of roughly 50 nm, while microtubules have a

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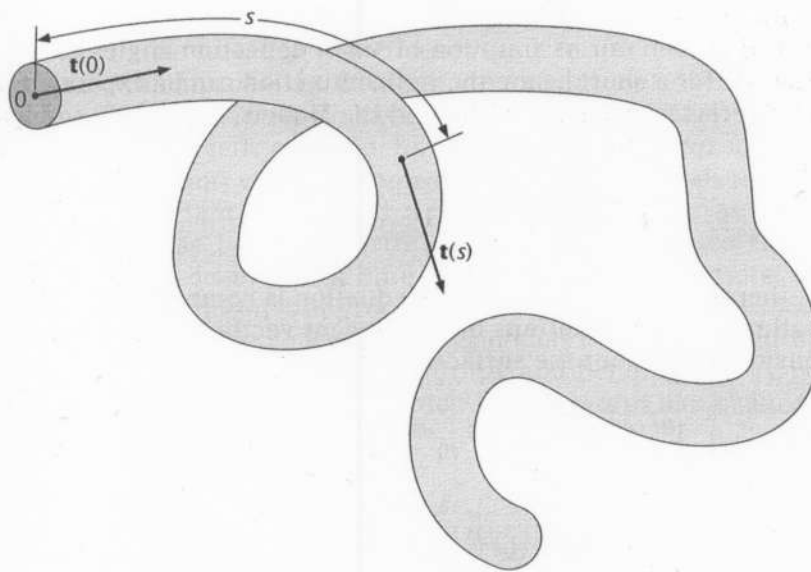


Figure 10.7 Tangent vectors of a fluctuating polymer chain. The parameter s measures the distance along the polymer. The tangent vector at contour length s is represented as $\mathbf{t}(s)$.

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persistence length of the order of 6 mm. To compute the persistence length, we average over all configurations of the polymer, with each such configuration assigned a Boltzmann weight that depends upon the bending energy. Before we plunge into the mathematics of the tangent-tangent correlation function

$$g(s) = \langle \mathbf{t}(s) \cdot \mathbf{t}(0) \rangle \quad (10.12)$$

it is useful to think about its limits. First, since the tangent vector is of unit length, $g(0) = 1$. On the other hand, for s much larger than the persistence length, we expect the two tangent vectors to be independent and $g(s) \rightarrow 0$. A simple function with these properties is an exponential function

$$g(s) = e^{-s/\xi_p} \quad (10.13)$$

where ξ_p is the persistence length. The mathematical proof that $g(s)$ is indeed exponential is somewhat subtle. It is based on the property that two adjacent segments of an elastic beam are buffeted by independent thermal forces and hence the tangent-tangent correlation function over both segments is the product of the individual ones, that is, $g(s_1 + s_2) = g(s_1)g(s_2)$; s_1 and s_2 are the lengths of the two segments (further details can be found in Nelson, 2004).

We seek a relation between the persistence length, as defined by eqn 10.13, and the flexural rigidity $K_{eff} = EI$ of an elastic beam. To this end we take a short beam of length $s \ll \xi_p$ and compute $g(s)$. For a short beam, thermal forces can only bend the beam slightly and the shape at any instant can be approximated by an arc of a circle of radius R . The energy of such a configuration is given by eqn 10.5, which can also be written as

$$E_{bend} = \frac{EI}{2s} \theta^2, \quad (10.14)$$

with $\theta = s/R$. Taking the tangent of one end of the beam to point along the z -direction, the tangent-tangent correlation function becomes

$$g(s) = \langle \cos \theta(s) \rangle, \quad (10.15)$$

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with $\theta(s)$ the angle between the tangent vector at the other end and the z -direction. Given our assumption of small deflection angles ($\theta(s) \ll 1$) appropriate for a short beam, the cosine function can be expanded into a Taylor series (see "The Math Behind the Models" on p. 193) to yield a simplified expression

$$g(s) = \left\langle 1 - \frac{\theta^2(s)}{2} \right\rangle. \quad (10.16)$$

The thermal average in the above equation is computed by summing over all possible orientations of the tangent vector at s , which in three dimensions traces out the surface of a unit sphere. Therefore

$$\langle \theta^2(s) \rangle = \frac{1}{Z} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \theta^2 e^{-(EI/2k_B Ts)\theta^2}, \quad (10.17)$$

where

$$Z = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta e^{-(EI/2k_B Ts)\theta^2}, \quad (10.18)$$

is the normalization factor (partition function). To simplify calculations we note that the integrand in eqn 10.17 differs from the integrand for Z by the factor θ^2 , which we can pull down from the Boltzmann factor in eqn 10.18 by differentiating with respect to E as introduced in Section 6.1 (p. 219). Then we can write

$$\langle \theta^2(s) \rangle = \frac{1}{Z} \left(-\frac{2k_B Ts}{I} \frac{\partial Z}{\partial E} \right) = -\frac{2k_B Ts}{I} \frac{\partial \ln Z}{\partial E}, \quad (10.19)$$

and all that remains to be calculated is the integral in eqn 10.18.

We compute the integral in question by making the substitution $\sin \theta \approx \theta$, valid for small angles, and by making use of a change of variables $u = (EI/2k_B Ts)\theta^2$

$$Z = \frac{2\pi k_B Ts}{EI} \int_0^\infty du e^{-u} = \frac{2\pi k_B Ts}{EI}. \quad (10.20)$$

Note that the upper integration bound for the variable u tends to infinity in the limit when s is much smaller than the persistence length. Using the result $\partial \ln Z / \partial E = -1/E$ in eqn 10.19, and substituting the value for $\langle \theta^2(s) \rangle$ obtained in this way in eqn 10.16, we arrive at the result

$$g(s) = 1 - \frac{k_B T}{EI} s. \quad (10.21)$$

Comparing this with eqn 10.13 in the $s \ll \xi_p$ limit we conclude

$$\xi_p = \frac{EI}{k_B T}. \quad (10.22)$$

Note that the scaling is precisely that found in eqn 10.11, which was based on a simple comparison of the thermal energy and the bending energy of an elastic beam bent into a circular arc of 1 radian. This result also allows us to rewrite the flexural rigidity as $EI = \xi_p k_B T$.

10.2.3 Elasticity and Entropy: The Worm-Like Chain

The Worm-Like Chain Model Accounts for Both the Elastic Energy and Entropy of Polymer Chains

So far in the book, we have used two different physical models to characterize the structure and free energy of polymers. In Chapter 8,

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