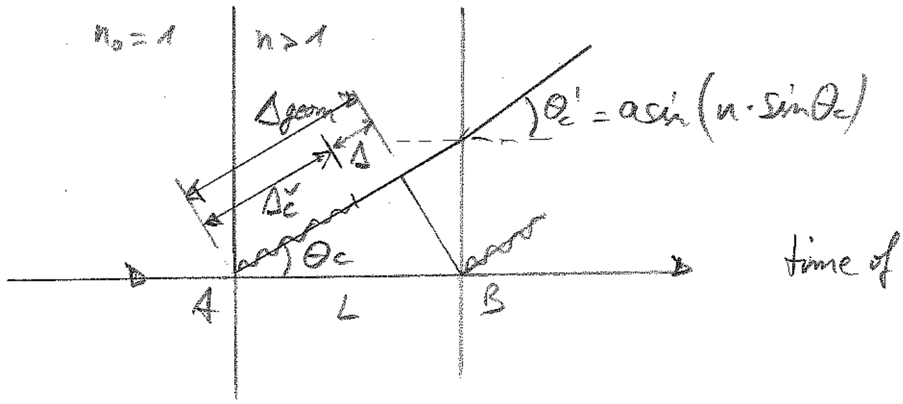
Derivation of Differential Flux of Transition Radiation (I)



Snellius law: $\frac{\sin\theta_c'}{\sin\theta} = \frac{n}{n_0} = n$

time of flight for particle from $A \to B$:

$$t = \frac{L}{v} = \frac{L}{\beta c}$$

$$\Delta_{geom} = L \cdot \cos\theta_c \quad , \quad \Delta_{\check{C}} = \frac{c}{n} \cdot t = \frac{c}{n} \cdot \frac{L}{\beta c} = \frac{L}{\beta n}$$

$$\Rightarrow \Delta = \Delta_{geom} - \Delta_{\check{C}} = L \cdot \left(\cos\theta_c - \frac{1}{\beta n}\right)$$

$\Rightarrow$ phase difference between Cherenkov light from A and B:

$$\Phi = \Delta \cdot \frac{2\pi}{\lambda} = \frac{2\pi L}{\lambda} \cdot \left(\cos\theta_c - \frac{1}{\beta n}\right) \quad (1)$$

- Flux of Cherenkov photons: $\frac{d^2 N_c}{dE\,dx} = \frac{\alpha z^2}{\hbar c} \cdot \sin^2\theta_c$ (2)
  - Substitute E by ω using $E = \hbar\omega$: $\frac{d^2 N_c}{d\omega\, dx} = \frac{d^2 N_c}{dE\, dx} \cdot \hbar$
  - Integrate over radiator length L: $\int_0^L dx \frac{d^2 N_c}{d\omega dx} = \frac{dN_c}{d\omega\, dx} \cdot L = \frac{dN_c}{d\omega}$
  - Differentiate by $d\Omega$, consider that radiation is into a cone of angle θ_c

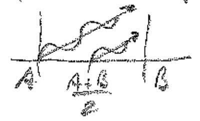
$$\hookrightarrow \frac{d^2 N_c}{d\omega d\Omega} = \frac{dN_c}{d\omega} \cdot \underbrace{\frac{1}{2\pi}}_{\text{due to azimuthal symmetry of cone around particle}} \underbrace{\delta\left(\cos\theta_c - \frac{1}{\beta n}\right)}_{\delta\text{-function takes care of angle } \theta_c \text{ for radiation}}$$

$$(2) \hookrightarrow \frac{d^2 N_c}{d\omega d\Omega} = \frac{\alpha z^2}{2\pi c} \cdot \sin^2\theta_c \cdot L \cdot \delta\left(\cos\theta_c - \frac{1}{\beta n}\right) \quad (3)$$

For $L \simeq \lambda$ the δ-function is replaced by Fraunhofer diffraction at a single slit:

$$\delta\left(\cos\theta_c - \frac{1}{\beta n}\right) \longrightarrow \frac{L}{\lambda} \cdot \frac{\sin^2 \Phi/2}{(\Phi/2)^2}$$

for a phase difference Φ as given in Eq. (1). Constructive interference requires a phase difference of $\frac{\Phi}{2} = 2\pi \cdot m$, $m \in \mathbb{N}$ for waves emitted from A and from the center at $\frac{A+B}{2}$:



$\Rightarrow$ relevant phase difference: $\Phi/2$

II

• ③ $\Rightarrow \frac{d^2 N_{TR}}{d\omega\, d\Omega} = \frac{\alpha z^2}{2\pi c} \sin^2\theta_c \cdot L \cdot \frac{L}{\lambda} \cdot \frac{\sin^2\frac{\phi}{2}}{(\phi/2)^2}$ ④

▸ approximate $\frac{\phi}{2} \overset{①}{=} \frac{1}{2}\frac{2\pi L}{\lambda}\left(\cos\theta_c - \frac{1}{\beta n}\right) \overset{\theta_c \text{ small}}{\approx} \frac{\pi L}{\lambda}\left(1 - \frac{\theta_c^2}{2} - \frac{1}{\beta n}\right)$

$$\left.\lambda = \frac{2\pi c}{\omega}\;,\; n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\right\} \Longrightarrow \approx \frac{1}{2}\frac{\omega L}{c}\left(1 - \frac{\theta_c^2}{2} - \frac{1}{\beta\sqrt{1-\omega_p^2/\omega^2}}\right)$$

$$\left.\frac{1}{\sqrt{1-\omega_p^2/\omega^2}} \overset{\omega_p \ll \omega}{\approx} 1 + \frac{\omega_p^2}{2\omega^2}\right\} \Longrightarrow \approx \frac{\omega L}{2c}\left(1 - \frac{\theta_c^2}{2} - \frac{1}{\beta}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)\right)$$

$$= \frac{\omega L}{2c}\left(1 - \frac{1}{\beta} - \frac{\theta_c^2}{2} - \frac{\omega_p^2}{2\beta\omega^2}\right)$$

$$= \frac{\omega L}{4\beta c}\left(2\beta - 2 - \beta\theta_c^2 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\left.-\frac{1}{\gamma^2} = \beta^2 - 1 = (\beta-1)(\beta+1) \overset{\beta\approx 1}{\approx} 2(\beta - 1)\right\} \Rightarrow = \frac{\omega L}{4\beta c}\left(-\frac{1}{\gamma^2} - \beta\theta_c^2 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\beta \approx 1 \Rightarrow \approx \frac{\omega L}{4c}\left(-\frac{1}{\gamma^2} - \theta_c^2 - \frac{\omega_p^2}{\omega^2}\right)$$

$$\Longrightarrow \frac{\phi}{2} \approx -\frac{\omega L}{4c}\left(\frac{1}{\gamma^2} + \theta_c^2 + \frac{\omega_p^2}{\omega^2}\right) \quad ⑤$$

• ⑤ in ④ $\Rightarrow$
$$\frac{\sin^2\frac{\phi}{2}}{(\phi/2)^2} \approx \frac{\sin^2\left[\frac{\omega L}{4c}\left(\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}\right)\right]}{\frac{\omega^2 L^2}{16c^2}\left(\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}\right)^2} \quad ⑥$$

for correct limit to vacuum case, i.e. $n \to 1 \Rightarrow \omega_p \to 0$, transition radiation should cease, i.e. $\frac{d^2 N_{TR}}{d\omega\, d\Omega} \xrightarrow{\omega_p \to 0} 0$

Need to correct ⑥ to account for this limit:

$$\frac{\sin^2\phi/2}{(\phi/2)^2} \longrightarrow \sin^2[\ldots\text{as in ⑥}] \cdot \frac{16c^2}{\omega^2 L^2}\underbrace{\left[\frac{1}{\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}} - \frac{1}{\theta_c^2 + \frac{1}{\gamma^2}}\right]^2}_{\to 0 \text{ for } \omega_p \to 0} \quad ⑦$$

• ⑦ in ④ and $\lambda = \frac{2\pi c}{\omega}$:

$$\Rightarrow \frac{d^2 N_{TR}}{d\omega\, d\Omega} \approx \frac{\alpha z^2}{2\pi c} \cdot \sin^2\theta_c \cdot \frac{L^2\omega}{2\pi c}\,\frac{16c^2}{\omega^2 L^2} \cdot \sin^2[\ldots\text{as in ⑥}] \cdot \left[\frac{1}{\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}} - \frac{1}{\theta_c^2 + \frac{1}{\gamma^2}}\right]^2$$

$$\left.\theta_c \text{ small}\right\} \Rightarrow \boxed{\frac{d^2 N_{TR}}{d\omega\, d\Omega} \approx \frac{\alpha z^2}{\pi^2 \omega} \cdot \theta_c^2 \cdot 4\sin^2\left[\frac{\omega L}{4c}\left(\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}\right)\right] \cdot \left[\frac{1}{\frac{\omega_p^2}{\omega^2} + \theta_c^2 + \frac{1}{\gamma^2}} - \frac{1}{\theta_c^2 + \frac{1}{\gamma^2}}\right]^2}$$