Exercises for the Lecture

Supersymmetry

Summer Semester 2016

Sheet 2

Exercise 1: Anti-symmetrized Gamma-matrices

a) Show that $\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = \mathrm{Id}_4$ and $\{\gamma^m, \gamma_5\} = 0$.
b) Confirm that for the anti-symmetrized gamma-matrices the following identities hold

$$\gamma^{mn} = \frac{i}{2}\epsilon^{mn}{}_{pq}\gamma_5\gamma^{pq}, \quad \gamma^{mnp} = -i\epsilon^{mnp}{}_q\gamma_5\gamma^q, \quad \gamma^{mnpq} = -i\epsilon^{mnpq}\gamma_5\,.$$

Exercise 2: Charge Conjugation and Chirality Projection

a) Proof the charge conjugation relations of the anti-symmetrized gamma-matrices (Defining relation: $C_\pm\gamma^m C_\pm^{-1} = \pm(\gamma^m)^T$)

$$C_\pm Id_4 C_\pm^{-1} = Id_4, \quad C_\pm\gamma^{mn}C_\pm^{-1} = -(\gamma^{mn})^T, \quad C_\pm\gamma_5\gamma^q C_\pm^{-1} = \mp(\gamma_5\gamma^q)^T, \quad C_\pm\gamma_5 C_\pm^{-1} = \gamma_5^T\,.$$

b) Show that $P_\pm$ is a projection.

Exercise 3: Dirac Conjugate

a) Derive the Lorentz-transformation of the Dirac conjugate spinor $\bar\Psi = \Psi^\dagger i\gamma_0$.
b) Show that $\bar\Psi\Psi$ is Hermitian and real.

Exercise 4: From Lorentz to Spin

a) Show that $\Lambda\gamma^m\Lambda^{-1} = \Lambda^m{}_n\gamma^n$ with $\Lambda = \exp(\frac{1}{2}B_{mn}\Sigma^{mn})$ and $\Lambda^m{}_n = \exp(B)^m{}_n$.
b) Deduce that the Spin group is the double cover of the Lorentz group.

Exercise 5: Pauli Matrices I

a) Confirm the Clifford property of the Pauli matrices, i.e. $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.
b) Check that the Pauli matrices generate the $su(2)$-algebra.
c) Show that $\epsilon(\sigma^i)^T\epsilon^T = -\sigma^i$.
d) Proof that

$$\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m = -2\eta^{mn}\mathrm{Id}_2 = \bar\sigma^m\sigma^n + \bar\sigma^n\sigma^m\,.$$

Exercise 6: Pauli Matrices II

a) Check the triple product identities of σ^m and $\bar{\sigma}^m$, i.e.

$$\sigma^m \bar{\sigma}^n \sigma^p = \eta^{mp}\sigma^n - \eta^{np}\sigma^m - \eta^{mn}\sigma^p + i\epsilon^{mnpq}\sigma_q \,,$$
$$\bar{\sigma}^m \sigma^n \bar{\sigma}^p = \eta^{mp}\bar{\sigma}^n - \eta^{np}\bar{\sigma}^m - \eta^{mn}\bar{\sigma}^p - i\epsilon^{mnpq}\bar{\sigma}_q \,.$$

b) Confirm the (anti-) commutation relations of the spin-generators $\sigma^{mn}, \bar{\sigma}^{pq}$.

c) Check the duality relations for the spin-generators

$$\frac{1}{2}\epsilon^{mnpq}\sigma_{pq} = -i\sigma^{mn}, \quad \frac{1}{2}\epsilon^{mnpq}\bar{\sigma}_{pq} = i\bar{\sigma}^{mn} \,.$$

d) Show that $\sigma^{mn}\epsilon$ and $\epsilon^T\bar{\sigma}^{mn}$ are symmetric matrices.

Exercise 7: Two-component Formalism

a) In the two-component formalism we can define a complex mass $m \in \mathbb{C}$, such that the mass term reads

$$m\chi\lambda + \bar{m}\bar{\chi}\bar{\lambda} \,,$$

where $\bar{m}$ is the complex conjugate of m. Write this mass term in the four-component formalism.

b) Show that $C_\pm$ have the right charge conjugation properties, for

$$C_\pm = \begin{pmatrix} \epsilon & 0 \\ 0 & \mp\epsilon^T \end{pmatrix} \,.$$

c) Show that

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

satisfies the Majorana condition.

For questions contact *sgreiner@mpp.mpg.de* ! Have fun!