3. Example of PEPs state: Kitaev's Toric code

[Kitaev 2003], [Kitaev 2009] is easy to read!

Simplest known model where ground state displays topological order.
Ground state on torus is four-fold degenerate — can be used to define a "topologically protected" qudit ...

- Square lattice (on 2D plane, or on torus)
- Spin 1/2 on each edge
- \( \hat{H} = -J_x \sum_s \hat{A}_s - J_y \sum_p \hat{B}_p \) (\( J_x, J_y > 0 \))
  
  sum over all stars \( s \), all plaquettes \( p \)

- \( \hat{A}_s = \prod_{i \in \text{star}(s)} \hat{\sigma}^z_i \)
- \( \hat{B}_p = \prod_{j \in \partial_p} \hat{\sigma}^x_j \)

[note: Kitaev means \( \hat{\sigma}^x \) for spins, \( \hat{\sigma}^z \) for plaquettes]

boundary of plaquette \( p \)

All terms in Hamiltonian commute

Easy to check: \([\hat{A}_s, \hat{B}_p]\) for all \( s, p \)

because all stars and plaquettes share an even number of edges (0 or 2);

minus signs from \( \hat{\sigma}^x \hat{\sigma}^z = -\hat{\sigma}^z \hat{\sigma}^x \) cancel: \((-1)^0 = (-1)^2 = 1\).

\( \Rightarrow \) All terms in \( \hat{H} \) commute \( \Rightarrow \) it should be adiabatic!

- Adopt \( \hat{\sigma}^z \) basis, label \( \hat{\sigma}^z \) eigenstates \( |\sigma_i\rangle, \sigma_i = \pm 1 \)

- Define "star flux": \( \hat{\omega}_s(\sigma) = \prod_{j \in \text{star}(s)} \hat{\sigma}^z_j \)

with eigenvalues \( \omega_s = \frac{1}{2} \)

If \( \omega_s = -1 \), there is a "woven" on star \( s \):
Ground state of toric code

The ground state satisfies: \( \hat{A}_s |g\rangle = |g\rangle \), \( \hat{B}_p |g\rangle = |g\rangle \) for all \( s, p \).

\[ \Rightarrow \text{need to maximize energy of all } \hat{A}_s, \hat{B}_p \text{ terms}. \]

\[ \hat{A}_s |g\rangle = |g\rangle \text{ only if } w_s = 1 \]

(all spins on star parallel, or two \( \uparrow \), two \( \downarrow \), for all plaquettes!)

Graphical notation: \( \uparrow \downarrow = \) \( \downarrow \uparrow = \cdots \)

Allowed vertex configurations: \( \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \) \( \checkmark \)

Not allowed: \( \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \) \( \text{(line ends)} \times \)

\[ \Rightarrow \text{ground state is "vortex-free", i.e. contains only closed loops} \]

\[ |g\rangle = \sum_{\{\bar{s}\}} c_{\bar{s}} |\bar{s}\rangle \quad \text{sum over all closed-loop configurations!} \]

\( \hat{B}_p \) flips all spins on plaquette, so \( \hat{B}_p |g\rangle = |g\rangle \)

Mnemonic: if \( \hat{B}_p (\bar{s}) = |\bar{s}'\rangle \), then \( c_{\bar{s}} = c_{\bar{s}'} \)

\[ \Rightarrow \text{Along each "orbit" of the action of plaquette operator, all } c_{\bar{s}} \text{ must be equal} \]

On plane:
Spin flips of plaquette operator are "ergodic", i.e. any closed-loop \( |\bar{s}\rangle \)
can be reached from any other one via series of \( \hat{B}_p \).

Hence, all \( c_{\bar{s}} \) must be equal:

\[ |g\rangle = \sum_{\{\bar{s}\}} c_{\bar{s}} |\bar{s}\rangle \quad \text{sum over all closed-loop configurations!} \]
**On torus:**

Let $l_1$ and $l_2$ be the “global loops” wrapping around surface of torus,
along the spin directions (i.e. between edges).

For given $l_1$ and $l_2$, define global loop functions:

\[ \hat{\gamma}_k(\vec{\sigma}) = \prod_{\vec{e} \in l_k} \hat{\sigma}^{\vec{e}}_i, \quad k = 1 \text{ or } 2 \]

Possible eigenvalues: $\hat{\gamma}_{l_1} = \pm 1, \quad \hat{\gamma}_{l_2} = \pm 1$

Any plaquette cuts $l_1$ and $l_2$ 0 or 2 times, i.e. flips an even number of spins, hence $[\hat{B}_p, \hat{\gamma}_{l_k}] = 0$

So, ground state $|g\rangle$ are also characterized by their $\hat{\gamma}_{l_k}$ eigenvalues:

\[ \hat{\gamma}_{l_1} |g\rangle = \hat{\gamma}_{l_1} |g, \omega_1, \omega_2\rangle, \quad \hat{\gamma}_{l_2} |g\rangle = \hat{\gamma}_{l_2} |g, \omega_1, \omega_2\rangle. \]

\[ \Rightarrow \text{There are 4 degenerate ground states!} \]

---

**Excitations for planar systems**

Excitations come in two varieties: (i) “electric charge”, (ii) “magnetic monopole”.

(i) Define “electric path operator”:

\[ \hat{W}^{(e)}_L = \prod_{j \in L} \hat{\sigma}^{x}_j \]

with $L$ = path from $s_1$ to $s_2$

\[ [\hat{W}^{(e)}_L, \hat{B}_p] = 0, \quad \hat{W}^{(e)}_L \hat{A}_s = \hat{A}_s \hat{W}^{(e)}_L, \quad \text{for } \{ s = s_1 \text{ or } s_2 \} \]

So, $|s_1, s_2\rangle = \hat{W}^{(e)}_L |g\rangle$ creates two “charges” at $s_1, s_2$ with energy $2J_e$ each.

(ii) Define “magnetic path operator”:

\[ \hat{W}^{(m)}_L = \prod_{j \in L^*} \hat{\sigma}^{y}_j \]

with $L^*$ = path on dual lattice, from $s_1$ to $s_2$

\[ [\hat{W}^{(m)}_L, \hat{A}_s] = 0, \quad \hat{W}^{(m)}_L \hat{B}_p = \hat{B}_p \hat{W}^{(m)}_L, \quad \text{for } \{ p = P_e \text{ or } P_m \} \]

So, $|s_1, s_2\rangle = \hat{W}^{(m)}_L |g\rangle$ creates two “vortices” at $p_1, p_2$, with energy $2J_m$ each.
PEPS representation of Toric code ground state [Verstraete 2008]  

$\sigma = \{ \uparrow = +1 \} \downarrow = -1$ 

$\Delta = \{ \sigma \}$ 

\[ \Delta \sigma \Delta \sigma \]  

[set auxiliary indices equal to physical indices] 

$D = 2$  

Summing over all PEPS on each vertex guarantees all possible loop encircling!  

PEPS - formulation generalizable to all "stringent models" [Gu 2009] 

which realize all non-chiral topological order in 2+1 D [Buerasflaper 2009].

---

Another example: Recasting AKLT loop state (RAL)  

2D square lattice, spin 1 per site: 

1) All loops must be fully packed and non-intersecting. 

2) Each site visited by exactly one loop. 

PEPS representation: [Li 2014] (Last author: Hang Hao Tu) 

\[ \alpha \rightarrow \beta \rightarrow \gamma \rightarrow \delta \rightarrow \epsilon \rightarrow \zeta \rightarrow \eta \rightarrow \kappa \rightarrow \lambda \rightarrow \mu \rightarrow \nu \rightarrow \xi \rightarrow \emptyset \] 

Procrater: Two spins -1/2 s on each site are projected to spin -1 other two "\$\$" to being "empty". 

\[ \mathcal{P}_i = (\sigma_{\uparrow} \sigma_{\uparrow} + \sigma_{\uparrow} \sigma_{\uparrow} + \cdots) \] 

(\text{same as for RVB state)
4. Some general comments about PEPS

(i) For translational invariance: periodically repeat a unit cell

(ii) PEPS are dense: any 2D state can be written as PEPS, at least if approximately large bond dimensions

(iii) 2D area law is satisfied: \( S_{AB} \sim O(e \log D) \)

(iv) PEPS can handle polynomially-decaying correlations (in contrast to 1D MPS)

(v) Exact contraction is \#P hard \( \Rightarrow \) contraction time \( \sim O(e^N) \).

\[ \begin{align*}
\#P \text{- hard class} & = \text{count number of solutions of NP-complete problem} \\
\text{NP-complete class} & = \text{problems that cannot be solved in polynomial time}
\end{align*} \]

\( \Leftrightarrow \) "non-deterministic polynomial"

(vi) No exact canonical form exists.

---

Why are contractions so hard?

Recall 1D situation:

Cheap contraction pattern:

Expensive contraction pattern:

In 2D, this is unavoidable:

\( \# \) of open indices is 2 throughout cost: \( O(d^3 D^3) \)

\( \# \) of open indices grow linearly, cost: \( O(d^4 D) \)

open indices: \( \begin{align*}
3 & \\
4 & \\
4 & \\
5 & \\
6 & \\
\end{align*} \)

just keep going...
No exact canonical form

Open MPS:

\[ \langle \Psi \rangle = \sum_a \langle a \rangle_c \langle b \rangle_e \]

Orthogonal state space enables efficient contractions:

\[ C = C, \quad D = D \]

Periodic boundary condition: "Loop MPS". Due to loop, it is not possible to define "left" and "right" pieces.

So, there are also no left/right canonical state spaces. For open/closed boundary conditions, PEPS is efficient/inefficient.

PEPS is full of loops! So this problem is even more acute!