**Tensor renormalization group (TRG)**

**Goal:** compute 2D contractions by coarse-graining RG scheme.
[as an alternative to transfer matrix schemes]

[Levin 2007], with Nave: original idea for TRG; 2D classical lattice models
local approach, optimizes truncation error only locally.

[Jiang 2008], with Weng, Xiang: adopt Levin-Nave idea to 2D quantum.
ground state projection using \( e^{-\beta H} \); "simple update" to do truncation.
TRG is used to compute expectation values (still only local approach).

[Xie 2009], Jiang, Chen, Weng, Xiang; [Zha 2010], Xie, Chen, Wei, Cai, Xiang
Second Renormalization (SRG), takes account renormalization of
environmental tensor; global approach. Reduces truncation error significantly.

[Xie 2012], Chen, Qin, Zhu, Yang, Xiang: different coarse-graining scheme,
using higher-order SVD, both local and global optimization schemes...

[Zha 2016], Xie, Xiang, Imada: coarse-graining on finite lattices...

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1. **TRG for 2D classical lattice models** [Levin 2007], with Nave

**Background:** express partition function as tensor network

**Goal:** contract it by coarse-graining scheme.

**Example:** 2D classical Ising, honeycomb lattice [Zha 2010]

**Hamiltonian:** \( H = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \), \( \sigma_i = \pm 1 \) Ising variable

**Lattice structure:**

![Honeycomb lattice diagram]

- Two sites per unit cell, \( a, b \)
- Three bond directions: \( x, y, z \)

**Partition function:**

\[
Z = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} e^{\beta \sigma_i \sigma_j} = \sum_{\{\Theta_i\}} \prod_{\langle i,j \rangle} \Theta_{ij}
\]
\[
\Theta_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{U_{ij}}{\lambda_i \lambda_j} V_{ij}^* = \sum_{i} \left( \frac{U_{i}}{\lambda_i} \right) \left( \frac{V_{i}}{\lambda_i} \right) = \varphi_{\sigma_{i,j}}^{\sigma_{i,j}}.
\]

Advantage of this representation: \( \sigma_i \) and \( \sigma_j \) dependence has been factorized, prior to \( \rho_{ij} \); matrix structure was introduced.

Group all \( \Lambda_{ij} \) that connect to site \( i \), and sum over \( \sigma_i \):

\[
T_x^{xyz} \ldots = \sum_{\sigma_i} \Lambda_{x \sigma_i \sigma_i}^{\sigma_i \sigma_i} \Lambda_{x \sigma_i \sigma_i}^{\sigma_i \sigma_i} \Lambda_{x \sigma_i \sigma_i}^{\sigma_i \sigma_i}.
\]

\[
T_x^{xyz} \ldots = \sum_{\sigma_j} \Lambda_{x \sigma_j \sigma_j}^{\sigma_j \sigma_j} \Lambda_{x \sigma_j \sigma_j}^{\sigma_j \sigma_j} \Lambda_{x \sigma_j \sigma_j}^{\sigma_j \sigma_j}.
\]

\[
Z = \sum_{\{\sigma_j \delta_{ij}\}} \Theta_{ij} = \sum_{\{\sigma_j \delta_{ij}\}} T_x^{i \sigma_{i,j} \sigma_{i,j}} T_y^{i \sigma_{i,j} \sigma_{i,j}} T_y^{i \sigma_{i,j} \sigma_{i,j}} = \text{tensor network}
\]

Cancel all n.n. bonds and sum over them.

All statistical physics models with short-ranged interactions can be written as tensor network models (i.e. \( Z = \text{Tr} \otimes \text{Tr} \otimes \text{Tr} \)).

Contracting out the tensor network by coarse-graining (Levin 2003).

"rewiring":

Iterate! Until \( T^a, T^b \) reach fixed point when \( T^a, T^b \)

Use these to compute free energy per spin: \( F = - \frac{1}{N_B} \ln Z \),

and from here the magnetization, etc.
2. Tensor renormalization group for quantum lattice models

[Chiang 2008], Weng, Xiang

Goal: compute ground state of 2D quantum lattice models

Strategy: iterative projection by $e^{-H_x}$, compress by simple update,
compute $\langle \Psi \vert \Psi \rangle$, $\langle 0 \vert 0 \rangle$ using TRG of Levin & Nave.

Model: $S = \frac{\epsilon}{2}$ Heisenberg

on honeycomb lattice.

Tensor network ansatz:

$$\langle \Psi \rangle = \text{Tr} \prod_i \prod_j A_i^x; A_i^y; A_i^z; B_j^x; B_j^y; B_j^z \vert \sigma_i; \sigma_j \rangle$$

Grand state projection via simple update

$$H = H_x + H_y + H_z \ (\text{having } x, y, \text{ or } z \text{-bonds})$$

Anyoni Trotter: $e^{-H_x} \approx e^{-TH_x} e^{-TH_y} e^{-\frac{1}{2} TH_z} + O(1^2)$

update $x$, $y$, and $z$ bonds sequentially using these three factors.

$$S = U \tilde{\Lambda}_x U^T, \quad \tilde{\Lambda}_x = \tilde{\lambda}_x^+ \tilde{\lambda}_x^- U, \quad \tilde{\sigma} = \tilde{\tau}_x^+ \tilde{\tau}_x^- U$$

"simple update": legs of $S$ contain $\frac{\tilde{\tau}_s}{\tilde{\lambda}_s}$ system \& environment

the four factors of $\tilde{\tau}_s$ account for environment in mean-field fashion.

(without them, procedure does not converge!)
• Similarly for y and z bonds
• Iterate simple update
• Start with $T = 10^{-3}$, gradually reduce it to $10^{-5}$
• Number of iterations: $10^5 - 10^6$

$<q|y> = \text{double layer tensor network}$

$T = \text{true net load indices at the end}$

$\text{last six sites}$

double $h$

 resultados

FIG. 5 (color online). The staggered magnetization $M(h)$ as a function of the staggered magnetic field, at different $D$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$E$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spin wave [12]</td>
<td>$-0.5489$</td>
<td>$0.24$</td>
</tr>
<tr>
<td>Series expansion [13]</td>
<td>$-0.5443$</td>
<td>$0.27$</td>
</tr>
<tr>
<td>Monte Carlo [14]</td>
<td>$-0.5450$</td>
<td>$0.22$</td>
</tr>
<tr>
<td>Ours $D = 8$</td>
<td>$-0.5506$</td>
<td>$0.21 \pm 0.01$</td>
</tr>
</tbody>
</table>
3. Second Renormalization (sRG) of Tensor Network State

[Xie 2009], more details: [Zhai 2011]

Goal: include influence of environment when doing update $\Rightarrow$ “global optimization”

Two applications: (i) 2D classical tensor network models
   (ii) 2D quantum ground states.

(i) Classical tensor network model: $Z = \text{Tr} \Pi \prod_{i} T_{i}^{a} T_{j}^{b} T_{3}^{c} T_{4}^{d}$

(blah, while hot assignments are)

(not constant in these figures)

removing:

SVD minimizes truncation error of $M$

However, we should minimize truncation error of $Z$.

Renormalize environment

Partition function:

$Z = \text{Tr} \ M M^{e}$

$= \sum_{ij} \text{M}_{ij} \text{M}_{jk} \text{M}_{ke} \delta_{i,j}$

Goal: minimize truncation error of $Z$

Strategy:

cheap “mean-field” approach

(i) compute $M^{e}$ — finite lattice
   $\Rightarrow$ more expensive forward/backward sRG

(ii) do SVD on “$M M^{e}$” — lets discuss (ii) first
Minimizing truncation error of $M^e$ [Zhao 2010], Sec. III.5

$$Z = \text{Tr} M M^e = \sum_{ijke} M_{ij} M^e_{ke}, \quad ij = i^e, M^e_{ke} = u_k e_k = u_k V_e^t$$

Treat row indices $ij$ and column indices $ke$ of $M$ "symmetrically".

Define $SVD$ of $M$:

$$D^2 = U_e \Lambda e V_e^t$$

$$\tilde{M} = \Lambda e^{1/2} V_e^t M U_e \Lambda e^{1/2} = \tilde{U} \Lambda \tilde{V}^t$$

Truncate $\tilde{U}$, hence $D^2$, is minimized.

Insert relation between $M$ and $\tilde{M}$:

$$M_{ij} e_k = u_k e_k \tilde{M} = u_k \Lambda e_k$$

$$= u_k \Lambda e_k U_e \Lambda e_k V_e^t = \sum_{i} S_{i} A_{i e} S_{k} A_{k e}$$

Computing environment tensor $M^e$: mean field approach [Xie 2009]

$M = U e \Lambda V^t$ defines the "singular bond vector" $\Lambda$, measures entanglement between two sites.

Cheap approach: "mean field approximation for environment", "simple update".

- Take $M^e = \sqrt{\Lambda} e_k \Lambda e_k$.
- Compute $\tilde{M}$, do SVD: $\tilde{M} = \tilde{U} \Lambda \tilde{V}^t$ (repeat for all bonds) new bond vector.
- Use new $\tilde{\Lambda}$'s to recalculate $M^e$, $\tilde{M}$, $\tilde{\Lambda}$, ...
- Iterate until convergence (typically 2 to 3 or energy more at critical point).
Even just a few environmental sites give a big improvement!

```
Commuting environment tensor $M^e$ : use TRG

Forward iteration:
(a) $\rightarrow$ (b) Rewire environment using data at iteration $n$:
\[
\sum T^n T = M^n = \lambda^n \Upsilon^n \Upsilon^t
\]
(b) $\rightarrow$ (c) Trace out triangles
(c) $\rightarrow$ (d) + (e) : identity new environment
(e) looks same as (a), only rotated by 90°.

Iteration relation:
$M^{n+1} = \sum M^{n, i} S S S S$
```

```
FIG. 10. (Color online) Relative errors of the free energy for the Ising model on a triangular lattice obtained by considering the second renormalization effect from four finite environment lattices which contains 4, 8, 14, and 22 sites, respectively. The configurations of these environments are shown in Fig. 9. The TRG result is also shown for comparison.
```

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\[
\delta f(T) = 1 - \frac{f(T)}{f_{\text{exact}}(T)}
\]
```
Computing Ising model at $M^\infty$: use TRG (cont.)

- Start with a very large but finite number of sites.
- Iterate until only 4 environmental sites are left:

  $\begin{array}{c}
  M^{(N)} = T^a T^b T^c T^d, \quad T = SSS \\
  \end{array}$

- "Backward iteration".
  - Start from current values of tensors $T^a, T^b$ and random vectors $\Lambda$.
  - Compute $M^{(N)}, N^{(N-1)}, N^{(N-2)}, \ldots$
  - All the way to $M^{(0)} = M^\infty = \text{desired result}$.

- Then compute $\tilde{M}, \tilde{\Lambda}, \tilde{M}$, and iterate until $\tilde{\Lambda}$'s have converged.

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**Results: SRG (2nd renormalization)**

Ising model on triangular lattice:

**Fig. 12.** (Color online) Comparison of the relative error of the free energy for the Ising model on triangular lattices obtained using TRG (red), the mean-field approximated SRG (blue), and the SRG (black) methods with $D_{\omega}=24$, respectively. The critical temperature is $T_c=4/\ln 3$.

**Fig. 13.** (Color online) The relative error of the free energy as a function of the truncation dimension $D_{\omega}$, for the Ising model on triangular lattices obtained using the TRG (black) and SRG (blue), respectively. $T=3.2$. 

"Mean field approach"
2nd renormalization for quantum ground state computation

Proceed analogously: optimize energy by projecting:

\[ S = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} = \frac{1}{\sqrt{D}} S \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \]

Trimming \( D \) to \( D' \) or reinstating \( a_x, a_y \)

The factors of \( \frac{1}{\sqrt{D}} \) on outside legs clarify that we really are just doing "mean-field" treatment of environment. For projection optimization, this is apparently OK, no SRG needed here.

Compute expectation values \( \langle \psi_1 \rangle, \langle \psi_0 \rangle \) using SRG, too!

Results

SRG yields more stable results than TRG!

FIG. 5 (color online). (a) The ground state energy per site \( E_0 \)
and staggered magnetization \( M_{\text{stag}} \) as functions of the
reefs of freedom \( D \) on honeycomb lattices.

[\( D = 6 \)]

Energy does not decrease with \( D \), but because SRG is not renormalized!