1. Entanglement Renormalization

[Osterloh 2009]

Real-space RG in 1D

If we cause a grain in a naive way, bond dimension grows exponentially.

But

- Fixed point of RG should be scale-invariant, the numerical resources needed should also remain the same from one step to the next.
- This would also be desirable, to reduce computational cost.
**Course-graining isometry**

**Course-graining map:** \( \Psi : \text{Vin} \rightarrow \text{Vout} \)

\[
\begin{align*}
\text{course-grained} & \quad \text{fine-grained} \\
\text{Vin} & \quad \text{Vout}
\end{align*}
\]

\[\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\end{pmatrix} = \begin{pmatrix}
\mu_1 \\
\mu_2 \\
\end{pmatrix}
\]

(*Vidal meso* contributes assignments of "in" and "out," because he views a MERA as a quantum circuit extending from top to bottom.)

**Isometry:**

\[
\mathbf{w}^+ \mathbf{w} = \mathbf{1}_{\text{Vin}}
\]

With indices:

\[
\begin{pmatrix}
\mu \\
\nu \\
\end{pmatrix}, \quad \sigma_1, \sigma_2, \sigma_3, \sigma_4, \mu = \delta_{\mu \nu}
\]

**Optimal choice for the states \( \mu \) (White's energy prescription):**

\[
\left( \rho_{\text{red}} \right)_{\sigma_1, \sigma_2, \sigma_3, \sigma_4} = \begin{pmatrix}
\delta_1 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 \\
0 & 0 & \delta_3 & 0 \\
0 & 0 & 0 & \delta_4
\end{pmatrix}
\]

**Diagonalize**

\[
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4
\end{pmatrix} = \begin{pmatrix}
\sigma_1 & 0 & 0 & 0 \\
0 & \sigma_2 & 0 & 0 \\
0 & 0 & \sigma_3 & 0 \\
0 & 0 & 0 & \sigma_4
\end{pmatrix}
\]

retain M largest eigenvalues, \( \mu = 1, \ldots, M \), that defines \( \mathbf{w} \)

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**Entanglement Renormalization: Disentanglers**

**Question:** can we reduce \( \mu_{\text{max}} \) needed to get good description of system

**Answer:** yes, by first reducing entanglement between block B and its environment before course-graining, using **disentanglers**.

**Disentanglers as unitary gates in span \( \{ | e_i \rangle \otimes | e_i \rangle \} \)**

\[
\mathbf{u}^+ \mathbf{u} = \mathbf{u} \mathbf{u}^+ = \mathbf{1}
\]

After disentangling,

\[
\left( \rho_{\text{red}} \right) = \begin{pmatrix}
\delta_1 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 \\
0 & 0 & \delta_3 & 0 \\
0 & 0 & 0 & \delta_4
\end{pmatrix}
\]

has eigenvalues \( \mu \) that decay more rapidly. So, fewer need to be retained:

\[
\tilde{\mu}_{\text{max}} < \mu_{\text{max}}
\]
Based on above considerations, Vidal proposed MERA ansatz for grand state:

\[ T = \log_2 L \]

Under an "RG step", local operators remain local (3 sites \( \rightarrow \) 3 sites).

This generates RG flow of operators (e.g. of Hamiltonian) and correlator:

\[ \omega \text{ and } \nu \text{ are found by variational minimization of } \langle \Psi | H | \Psi \rangle \]

**Results: Block entropy**

1D quantum Ising model with transverse field: \[ H = \sum_i \sigma_i^x \sigma_i^{x'} + h_z \sum_i \sigma_i^z \]

Entanglement entropy between block \( B \) of \( L \) adjacent spins and rest of lattice:

\[ S(L) = -\sum \mu \log_2 \mu \]

at criticality: \[ \approx \frac{1}{6} \log_2 L \]

without renormalization:

\[ S(L) \propto \text{const.} \]

with entanglement renormalization:

\[ S(L) \propto \text{const.} \]

**FIG. 3** (color online). Scaling of the entropy of entanglement in 1D quantum Ising model with transverse magnetic field. Up: in a critical lattice \( [h = 1 \text{ in Eq. (10)}] \), the unrenormalized entanglement of the block scales with the block size \( L \) according to Eq. (11). Instead, the renormalized entanglement remains constant under successive RG transformations, as a clear manifestation of scale invariance. Line (i) corresponds to using disentanglers only in the first RG transformation. Line (ii) corresponds to using disentanglers only in the first and second RG transformation. Down: in a noncritical lattice \( [h = 1.001 \text{ in Eq. (10)}] \), the unrenormalized entanglement scales roughly as in the critical case until it saturates (a) for block sizes comparable to the correlation length. Beyond that length scale, the renormalized entanglement vanishes (b) and the system becomes effectively unentangled.
Results: Spectrum of reduced density matrix

Eigenvalues of reduced density matrix of block of size $L$:

We need to keep all eigenvalues above threshold, set by

$$1 - \sum_{\mu=1}^L \lambda^\mu \leq \varepsilon$$

without renormalization:

$$\lambda_{\text{max}} \sim 2^{-L}$$

with renormalization:

spectrum is invariant under RG flows.

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2. MERA - Technical details

Quantum circuit interpretation:

MERA = quantum circuit

- $\mathcal{V}$ = local state space of one site
- Hilbert space of physical lattice: $\mathcal{V}_L = \mathcal{V}^\otimes N$
- $N$ cross (blue line)
  - each described by a Hilbert space $\mathcal{V}$
- Input state of each unit: fixed state $|\psi\rangle$
- MERA maps $|\psi\rangle^N$ onto physical state $|\psi\rangle$
  - using purely unitary gates.
- Unitaries mingling $|\psi\rangle$ are by can be thought of as isometries
MERA - Analysis is not unique

Binary 1D MERA:

\[ k=1,2,\ldots,\chi_T \]

\[ U_t \]

\[ \tau \]

\[ \tau^{+1} \]

\[ \tau^{+2} \]

\[ O_{t-1}^{[r,r^+1,r^+2]} \]

\[ L \]

\[ L_{t-1} \]

\[ \text{RG transformation} \]

- Local observables and up vertex supported on: 3 sites on 2 sites
- Local observables: \( O(\chi^4) \) \( O(\chi^8) \)
- two-point correlators: very expensive.

Ternary 1D MERA:

\[ k=1,2,\ldots,\chi_T \]

\[ U_t \]

\[ \tau \]

\[ \tau^{+1} \]

\[ O_{t-1}^{[r,r^+1]} \]

\[ L \]

\[ L_{t-1} \]

\[ \text{RG transformation} \]

- Local observables: \( O(\chi^8) \)
- two-point correlators: very expensive.

Examples of 2D MERA

(i) 2x2 MERA scheme for 2D square lattice

Original Lattice

Apply Disentanglers

Coarse-grained Lattice

- cost for local observables: \( O(\chi^{16}) \)

(ii) 3x3 MERA scheme for 2D square lattice

Original Lattice

Apply Disentanglers

Coarse-grained Lattice

- cost for local observables: \( O(\chi^{28}) \)
Ascending superoperators (needed to compute expectation values)

Renormalize operators using "ascending superoperators", which describe "operator flow" under RG step:

These maps to renormalize two-site operator with ternary MERA:

Ascending superoperators:

\[ (a') o_t = A_L(o_{t-1}) \]
\[ (b') o_t = A_C(o_{t-1}) \]
\[ (c') o_t = A_R(o_{t-1}) \]

Then average:

\[ \bar{A} = \frac{1}{3} (A_1 + A_2 + A_3) \]

Expectation values A operators

\[ \langle \delta^{(r+1)} \rangle_4 = \langle \psi | \delta^{(r+1)} | \psi \rangle \]

Expected value of a local observable:

(ii)

All tensors lying outside the "causal cone" (purple) can be contracted trivially

\[ \Rightarrow \text{ MERA is contractible at low cost!} \]

(Also in 2D)
Performing remaining contractions amounts to repeated application of ascending superoperators.

Two-site operator

Two-point correlator:

Note: translational invariance is broken: not all sites are equivalent: some sites have very near causal cores.
Energy optimization

\[ E(\omega) = \text{Tr} \left( \sum_s M_s \omega_s \omega_s^\dagger N_s \right) + c, \quad \text{quadratic optimization with constant: } c = 1 \]

There is no known algorithm for this; one iterative approach:

- Linearize cost function (treat \( \omega \) independent of \( \omega \)):
  \[ E(\omega) = \text{Tr} \left( \omega Y_0 \right), \quad \text{with } Y_0 = \sum_s M_s \omega_s^\dagger N_s \]

- SVD on \( Y = V S W^\dagger \), choose \( \omega = -W V^\dagger \)
  \[ \text{Then } E(\omega) = \text{Tr} \left( -W V^\dagger V S W^\dagger \right) = -\text{Tr}(S) = -\sum_s s_\lambda \]

Environmental tensors

A disentangler can occur in 6 different positions relative to a 2-site gate, so its environment is constructed by averaging over all 6 possibilities.

Environments of an isometry:

\[ Y_\omega = \frac{1}{6} \sum_{i=1}^{6} Y_i \]

Similarly for disentangler:

\[ Y_\omega = \frac{1}{6} \sum_{i=1}^{6} Y_i \]

Environments of a disentangler:
FIG. 24. (Color online) (Top) The energy error of the MERA approximations to the ground state of the infinite Ising model, as compared against exact analytic values, is plotted both for different transverse magnetic field strengths and different values of the MERA refinement parameter $\chi$. The finite-correlation-range algorithm (with at most $T=5$ levels) was used for noncritical ground states, while the scale-invariant MERA was used for simulations at the critical point. It is seen that representing the ground state is most computationally demanding at the critical point, although even at criticality the MERA approximates the ground state to between five digits of accuracy ($\chi=4$) and ten digits of accuracy ($\chi=22$). (Bottom) Scale-invariant MERA are used to compute the ground states of infinite, critical, 1D spin chains of Eqs. (75)-(78) for several values of $\chi$. In all instances one observes a roughly exponential convergence in energy over a wide range of values for $\chi$ as indicated by trend lines (dashed line). Energy errors for Ising, XX, and Heisenberg models are taken relative to the analytic values for ground energy, while energy errors presented for the Potts model are taken relative to the energy of a $\chi=22$ simulation.

\[ e^{-\text{exponential converges with } \chi} \]
3. Tensor Network Renormalization [Evenbly 2015] [Evenbly 2017]

**Goal:** improve tensor renormalization group (TRG) for classical partition function or for Euclidean path integral by introducing disentanglers.

Partition function can be written as tensor network (see Lecture 24).

\[ Z = \text{Tr} (\bigotimes_{x=1}^{N} A^{(x)}) \rightarrow \text{Tr} (\bigotimes_{x=1}^{N/4} A^{(x)}) \rightarrow \text{Tr} (\bigotimes_{x=1}^{N/4^2} A^{(x)}) \ldots \]

Strategy: coarse-grain successively, so that after \( S \) steps,

\[ Z = \text{Tr} (\bigotimes_{x=1}^{N/4^S} A^{(x)}) \] describes physics at length scale \( a^S \)

where \( a \) = lattice constant.

After \( S = \log_4(N) \) steps, partition function is just a single tensor:

\[ Z = \text{Tr} A^{(s)} \] which can be evaluated.
Using disentanglers to remove irrelevant local correlations

(a) original lattice
(b) tensor network for $z$
(c) Insert disentanglers between two pairs of sites: $u, v$

Then reduce dimensions using isometries $\Downarrow$ as

(b) Find $U, W$ by minimizing truncation error.

Disentangler removes irrelevant local correlations,

TNR scheme

(a) disentangle and truncate
(b) define coarse-grained lattice
(c) SUD and truncate, as in usual TNR, to restore lattice geometry
(d) repeat until to define coarse-grained lattice.

(no 2nd-RG step used here ...)
FIG. 3 (color online). Benchmark results for the square lattice Ising model on a lattice with $2^{39}$ spins. (a) Relative error in the free energy per site $\delta f$ at the critical temperature $T_C$ comparing the TRG and TNR over a range of bond dimensions $\chi$. The TRG errors fit $\delta f \propto \chi^{-0.2}$ (the inset displays them using log-log axes), while TNR errors fit $\delta f \propto \exp(-0.305\chi)$. Extrapolation suggests that the TRG would require bond dimension $\chi \approx 750$ to match the accuracy of the $\chi = 42$ TNR result. (b) Spontaneous magnetization $M(T)$ obtained with TNR with $\chi = 6$ [30]. Even very close to the critical temperature, $T = 0.9994T_C$, the magnetization $M \approx 0.48$ is reproduced to within 1% accuracy.

FIG. 4 (color online). (a) Singular values $\lambda_s$ of the matrix $[A^{(s)}]_{\mu=0}^{\mu=0}$ obtained after $s$ RG steps [31] using TNR (the filled circles) or the TRG (the empty circles) for the 2D Ising model at critical temperature $T_C$. (b) Singular values for $T = 1.1T_C$. (c) Plot of the von Neumann entropy $-\sum \lambda_s \log(\lambda_s)$ of the (normalized) singular values of tensors $[A^{(s)}]_{\mu=0}^{\mu=0}$ obtained with the TRG (the empty circles) or TNR (the filled circles).