**MERA: Multiscale Entanglement Renormalization**

Goal: reduce entanglement generated when miniaturizing quantum degrees of freedom at each RG step.

Strategy: Use "disentanglers", i.e., isometries that reduce entanglement between blocks.

[Vidal 2007] Original idea for MERA; application to tensor fields. Toy model.
[Vidal 2008] Shows that contractions in MERA can be computed efficiently; ent: O(N).
[Koenig 2009] of matter; shows Kitaev toric code ground state is unique MERA.
[Ritzi 2008] L-MERA; optimization in time evolution.
[Cincio 2008] MERA in 2D: Quantum Ising model.
[EventEly 2009a] MERA in 2D: Anyon chain ends from O(N^2) to O(N).
[EventEly 2010] Fermionic MERA.
[EventEly 2015a] MERA: Tensor network renormalization, applied to Te Fat, yields MERA.

1. Entanglement Renormalization  

Read your RG in 1D.
**Course-graining identity**

Course-graining map: \( \omega : V_{\text{in}} \rightarrow V_{\text{out}} \)

\[ \omega^\dagger \omega = \mathbb{I}_{V_{\text{in}}} \]

with indices:

\[ \sum_{\mu, \mu'} \omega^\dagger_{\mu, \mu'} \omega_{\mu', \mu} = \delta_{\mu, \mu'} \]

**Intanglement Renormalization: Disentanglers**

Idea: use disentanglers to remove correlations between block B and environment.

Disentanglers: unitary gates \( u \in \text{span} \{ |e_1 \rangle \otimes |e_2 \rangle \} \), \( u^\dagger u = uu^\dagger = \mathbb{I} \)

\[ \langle \text{prod} \rangle_{\tilde{\xi}, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3} = \]

\[ \tilde{m}_{\text{max}} < m_{\text{max}} \]
Based on above considerations, Vidal proposed MERA ansatz for ground state:

\[ T = \log_{2} L \]

Under an “RG step”, local operators remain local (3 sites \( \Rightarrow 2 \) sites).
This generates RG flow of operators (e.g. of Hamiltonian) and correlators.

\[ \omega \text{ and } \nu \text{ are found by minimal minimization of } \langle 41H14 \rangle \]

**Results: Block entropy**

1D quantum Ising model with transverse field: \[ H = \sum_{x} \sigma_{x}^{i} \sigma_{x+1}^{i} + J \sum_{i} \sigma_{i}^{z} \]

Entanglement entropy between block \( B \) of \( L \) adjacent spins and rest of lattice:

\[ E = \frac{1}{6} \log L + k \]

FIG. 3 (color online). Scaling of the entropy of entanglement in 1D quantum Ising model with transverse magnetic field. Up: in a critical lattice \( [h = 1 \text{ in Eq. (10)}] \), the unrenormalized entanglement of the block scales with the block size \( L \) according to Eq. (11). Instead, the renormalized entanglement remains constant under successive RG transformations, as a clear manifestation of scale invariance. Line (i) corresponds to using disentanglers only in the first RG transformation. Line (ii) corresponds to using disentanglers only in the first and second RG transformation. Down: in a noncritical lattice \( [h = 1.001 \text{ in Eq. (10)}] \), the unrenormalized entanglement scales roughly as in the critical case until it saturates (a) for block sizes comparable to the correlation length. Beyond that length scale, the renormalized entanglement vanishes (b) and the system becomes effectively unentangled.
Results: Spectrum of reduced density matrix

Eigenvalues of reduced density matrix of block of size $L$:

We need to keep all eigenvalues above threshold, set by

$$1 - \sum_{\mu=1}^{L^2} \rho_{\mu} \leq \varepsilon$$

without renormalization:

$$\max \rho \sim \varepsilon^{-1} \sim L$$

with renormalization:

spectrum is invariant under RG flows.

FIG. 4 (color online). Spectrum of the reduced density matrix of a spin block. Up: as the size $L$ of the spin increases, the number $m$ of eigenvalues $\{p_i\}$ required to achieve a given accuracy $\varepsilon$, see Eq. (5), also increases. In particular, $m$ grows roughly exponentially in the number $\tau = \log_2 L$ of RG transformations. The spectrum resulting from applying disentanglers leads to a significantly smaller $m$ invariant along successive RG transformations. Down: spectrum of the reduced density matrix of $2^\tau$ spins immediately before and after using the disentanglers at the $\tau^{th}$ coarse-graining step. These spectra are essentially independent of the value of $\tau = 1, \cdots, 14$.

2. MERA - Technical details

Quantum circuit implementation:

$$V = \text{local unitary of size } k$$

$N$ input states $|0\rangle$

$N$ modes:

MERA maps:

$$|0\rangle^n \rightarrow 145 \in V_L$$

Quantum Circuit:

Input

(unitary isometry)
MERA - Theory is not unique

Binary 1D MERA:

Ternary 1D MERA:

Local observables: \( 3 \text{sites} \rightarrow 3 \text{sites} \)

Cost for \( \langle \phi \rangle \): \( O(\lambda^7) \)

two point correlator: very expensive

Examples of 2D MERA

(i) 2x2 MERA scheme for 2D square lattice

(ii) 3x3 MERA scheme for 2D square lattice

Original Lattice

Apply Disentanglers

Coarse-grained Lattice

Original Lattice

Apply Disentanglers

Coarse-grained Lattice

\( O(\lambda^{14}) \)

\( O(\lambda^{28}) \)
**Ascending superoperators** (needed to compute expectation values)

Renormalize operators using "ascending superoperators", which describe "operator flow" under RG step:

Those maps to renormalize two-site operator with ternary MERA:

**Ascending superoperators**:

\[
(a') o_t = A_L(o_{t-1}) \\
(b') o_t = A_C(o_{t-1}) \\
(c') o_t = A_R(o_{t-1})
\]

Then average:

\[
\overline{A} = \frac{1}{3}(A_1 + A_2 + A_3)
\]

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**Expectation values A operators**

Expected value of a local observable:

(i)

(ii)

All tensors lying outside the "causal core" (purple) can be contracted trivially:

\[
\begin{array}{c}
\text{causal zone} = N \\
\end{array}
\]

\[
\begin{array}{c}
=\quad || \\
\end{array}
\]
Performing remaining contractions amounts to repeated application of ascending superoperators.

Two-site operators

Two-point correlator:

Note: translational invariance is broken: not all sites are equivalent: some sites have very narrow causal cones:

\[
\text{Two-site operators}
\]
Energy optimization

(i) Energy: $E = \text{tr}(HP)$

(ii) Isometry Environment: $Y$

$$E(w) = \text{Tr} \sum_i (w_i N_i w_i^* N_i)$$

**Quadratic form for $w$**

**Constraint:** $w^* w = 1$

**Linear:** $E(w) = \text{Tr} (w Y w)$

**SVD:** $Y = U S W^*$

where: $w = -W V^+$

$$E(w) = \text{Tr} (-W V^* U S W^*) = -\text{Tr} S = -\sum_i s_i$$

Environmental tensors

A disentangler can occur in 6 different positions relative to a 2-site gate, so its environment is constructed by averaging over all 6 possibilities.

Environments of an isometry:

$$Y_{w} = \sum_{i=1}^{6} Y_{w}^{i}$$

Similarly for disentangler:

Environments of a disentangler:

$$Y_{w} = \sum_{i=1}^{6} Y_{w}^{i}$$
FIG. 24. (Color online) (Top) The energy error of the MERA approximations to the ground state of the infinite Ising model, as compared against exact analytic values, is plotted both for different transverse magnetic field strengths and different values of the MERA refinement parameter $\chi$. The finite-correlation-range algorithm (with at most $T=5$ levels) was used for noncritical ground states, while the scale-invariant MERA was used for simulations at the critical point. It is seen that representing the ground state is most computationally demanding at the critical point, although even at criticality the MERA approximates the ground state to between five digits of accuracy ($\chi=4$) and ten digits of accuracy ($\chi=22$). (Bottom) Scale-invariant MERA are used to compute the ground states of infinite, critical, 1D spin chains of Eqs. (75)-(78) for several values of $\chi$. In all instances one observes a roughly exponential convergence in energy over a wide range of values for $\chi$ as indicated by trend lines (dashed line). Energy errors for Ising, XX, and Heisenberg models are taken relative to the analytic values for ground energy, while energy errors presented for the Potts model are taken relative to the energy of a $\chi=22$ simulation.

Exponential convergence with $\chi$
MERA results

3. Tensor Network renormalization [Evenbly 2015] [Evenbly 2017]

Goal: improve tensor renormalization group (TRG) for classical partition function or for Euclidean path integral by introducing disentanglers.

Partition function can be written as tensor network (see Lecture 24).

\[ Z = \text{Tr} (\otimes_{x=1}^{N} A^{(x)}) \rightarrow \text{Tr} (\otimes_{x=1}^{N/4} A^{(x)}) \rightarrow \text{Tr} (\otimes_{x=1}^{N/16} A^{(x)}) \ldots \]

Strategy: coarse-grain successively, so that after 5 steps,

\[ Z = \text{Tr} (\otimes_{x=1}^{N/32} A^{(s)}) \text{ describes physics at length scale } a^5 \]

where \( a \) = lattice constant.

After \( s = \log_4(N) \) steps, partition function is just a single tensor:

\[ Z = \text{Tr} A^{(s)} \text{ which can be evaluated.} \]
**Using disentanglers to remove irrelevant local correlations**

(a) original lattice

(b) tensor network for \( z \).

(c) Insert disentanglers between two pairs of sites:

Then reduce dimensions using isometries as

(d) Find \( U, W \) by minimizing truncation error.

Disentangler removes irrelevant local correlations.

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**TNR scheme**

(a) disentangle and truncate

(b) define coarse-grained lattice

(c) SUD and truncate, as in usual TNR, to restore lattice geometry

(d) contract out, to define coarse-grained lattice.

(no 2nd-RG step used here ... )
FIG. 3 (color online). Benchmark results for the square lattice Ising model on a lattice with $2^{39}$ spins. (a) Relative error in the free energy per site of $\delta f$ at the critical temperature $T_c$, comparing the TRG and TNR over a range of bond dimensions $\chi$. The TRG errors fit $\delta f \propto \chi^{-0.2}$ (the inset displays them using log-log axes), while TNR errors fit $\delta f \propto \exp(-0.305\chi)$. Extrapolation suggests that the TRG would require bond dimension $\chi \approx 750$ to match the accuracy of the $\chi = 42$ TNR result. (b) Spontaneous magnetization $M(T)$ obtained with TNR with $\chi = 6$ [30]. Even very close to the critical temperature, $T = 0.9994T_c$, the magnetization $M \approx 0.48$ is reproduced to within 1% accuracy.

FIG. 4 (color online). (a) Singular values $\lambda_s$ of the matrix $[A^{(s)}]_{i,j}$ obtained after $s$ RG steps [31] using TNR (the filled circles) or the TRG (the empty circles) for the 2D Ising model at critical temperature $T_c$. (b) Singular values for $T = 1.1T_c$. (c) Plot of the von Neumann entropy $-\sum_s \lambda_s \log(\lambda_s)$ of the (normalized) singular values of tensors $[A^{(s)}]_{i,j,m}$ obtained with the TRG (the empty circles) or TNR (the filled circles).