Hydrodynamics Sommersemester 2019

Problem sheet 1 - Solutions

Problem 1.1:

1.1.1:

$$\begin{aligned}
\vec{\nabla}p &= \rho\vec{g} \qquad | \qquad \vec{g} = -g\vec{e}_z \\
\frac{dp}{dz} &= -\rho g \qquad | \qquad \text{polytropic:} \frac{p}{\rho^n} = const. = \frac{p_0}{\rho_0^n} \\
\frac{dp}{p^{\frac{1}{n}}} &= -g\frac{\rho_0}{p_0^{\frac{1}{n}}}dz \\
\int_{p(0)}^{p(z)} \frac{dp}{p^{\frac{1}{n}}} &= -g\frac{\rho_0}{p_0^{\frac{1}{n}}}\int_0^z dz' \qquad | \qquad p(0) = p_0 \\
\frac{n}{n-1}[p(z)^{\frac{n-1}{n}} - p_0{}^{\frac{n-1}{n}}] &= -g\frac{\rho_0}{p_0^{\frac{1}{n}}}z \qquad | \qquad +p_0{}^{\frac{n-1}{n}} \\
p(z)^{\frac{n-1}{n}} &= p_0{}^{\frac{n-1}{n}} - \frac{n-1}{n}g\frac{\rho_0}{p_0^{\frac{1}{n}}}z \\
p(z)^{\frac{n-1}{n}} &= p_0{}^{\frac{n-1}{n}}[1 - \frac{n-1}{n}\frac{\rho_0}{p_0}gz] \qquad | \qquad [\,]^{\frac{n}{n-1}} \\
p(z) &= p_0[1 - \frac{n-1}{n}\frac{\rho_0}{p_0}gz]^{\frac{n}{n-1}} \\
&\square
\end{aligned}$$

For the height h of the atmosphere, we have: $p(h) \stackrel{!}{=} 0$

$$\begin{aligned}
&\Rightarrow \quad 1 - \frac{n-1}{n}\frac{\rho_0}{p_0}gh \stackrel{!}{=} 0 \\
&\Rightarrow \quad h = \frac{n}{n-1}\frac{p_0}{\rho_0 g}
\end{aligned}$$

1.1.2:

to be shown:

$$\lim_{n\to 1} p_0[1 - \frac{n-1}{n}\frac{\rho_0}{p_0}gz]^{\frac{n}{n-1}} \stackrel{!}{=} p_0 e^{-\frac{\rho_0}{p_0}gz}$$

$$\text{Substitution:} \quad 1) \quad x = \frac{\rho_0}{p_0}gz \qquad 2) \quad \alpha = \frac{n}{n-1}$$

now we have to show:

$$\lim_{\alpha\to\infty}[1 - \frac{x}{\alpha}]^\alpha \stackrel{!}{=} e^{-x}$$

Since ln is continuous at 1, we have:

If $\lim_{\alpha\to\infty} \alpha \ln[1 - \frac{x}{\alpha}]$ exists, then $\lim_{\alpha\to\infty}[1 - \frac{x}{\alpha}]^\alpha$ exists as well and

$$\lim_{\alpha\to\infty} \alpha \ln[1-\frac{x}{\alpha}] \;=\; \ln(\lim_{\alpha\to\infty}[1-\frac{x}{\alpha}]^{\alpha})$$

Consider:

$$\lim_{\alpha\to\infty} \alpha \ln[1-\frac{x}{\alpha}] \;=\; \lim_{\alpha\to\infty} \frac{\ln[1-\frac{x}{\alpha}]}{\alpha^{-1}}$$

Using the rule of L'Hospital ('0 divided by 0') we obtain

$$\begin{aligned}
& \lim_{\alpha\to\infty} \frac{\ln[1-\frac{x}{\alpha}]}{\alpha^{-1}} = \lim_{\alpha\to\infty} \frac{\frac{d\left(\ln[1-\frac{x}{\alpha}]\right)}{d\alpha}}{\frac{d\left(\alpha^{-1}\right)}{d\alpha}} \\
\Rightarrow \quad & \lim_{\alpha\to\infty} \frac{\ln[1-\frac{x}{\alpha}]}{\alpha^{-1}} = \lim_{\alpha\to\infty} \frac{\frac{1}{1-\frac{x}{\alpha}}\left(\frac{x}{\alpha^2}\right)}{\frac{-1}{\alpha^2}} = \lim_{\alpha\to\infty} \frac{-x}{1-\frac{x}{\alpha}} = -x \\
\Rightarrow \quad & \ln(\lim_{\alpha\to\infty}[1-\frac{x}{\alpha}]^{\alpha}) = -x \\
\Rightarrow \quad & \lim_{\alpha\to\infty}[1-\frac{x}{\alpha}]^{\alpha} = e^{-x} \\
& \square
\end{aligned}$$

1.1.3:

$$\Psi = -\frac{GM}{r}; \qquad F = -\vec{\nabla}\Psi$$

radial:

$$\begin{aligned}
\frac{dp}{dr} &= -\rho\frac{GM}{r^2} \quad ; \qquad \Rightarrow g = \frac{GM}{R_E^2} \\
\int_{p(R_E)}^{p(r)} \frac{dp}{p^{\frac{1}{n}}} &= GM\frac{\rho_0}{p_0^{\frac{1}{n}}}\int_{R_e}^{r}\frac{1}{r'^2}dr' \\
p(r) &= p_0[1-\frac{n-1}{n}\frac{\rho_0}{p_0}GM(\frac{1}{R_E}-\frac{1}{r})]^{\frac{n}{n-1}}
\end{aligned}$$

Use $g = \frac{GM}{R_E^2}$:

$$p(r) \;=\; p_0[1-\frac{n-1}{n}\frac{\rho_0}{p_0}g(R_E-\frac{R_E^2}{r})]^{\frac{n}{n-1}}$$

Write $r = R_E + z$:

$$\begin{aligned}
p(z) &= p_0[1-\frac{n-1}{n}\frac{\rho_0}{p_0}g(R_E-\frac{R_E^2}{R_E+z})]^{\frac{n}{n-1}} \\
p(z) &= p_0[1-\frac{n-1}{n}\frac{\rho_0}{p_0}gz(\frac{R_E}{R_E+z})]^{\frac{n}{n-1}}
\end{aligned}$$

For the height h of the atmosphere, we obtain: $p(h) \overset{!}{=} 0$

$$\Rightarrow \qquad 1 - \frac{n-1}{n}\frac{\rho_0}{p_0}gh(\frac{R_E}{R_E+h}) \overset{!}{=} 0$$

For $h \ll R_E$ we have $\frac{R_E}{R_E+h} \approx 1$, so that:

$$\Rightarrow \qquad h \approx \frac{n}{n-1}\frac{p_0}{\rho_0 g}$$

□

Problem 1.2:

Given:

$$\vec{v} = -ay\vec{e}_x + bx\vec{e}_y$$

$$\rho = \hat{\rho} = const.$$

We use:

$$\partial_x := \frac{\partial}{\partial x}; \quad \partial_y := \frac{\partial}{\partial y}; \ldots$$

Stationary equation of continuity is fulfilled:

$$\vec{\nabla}(\rho\vec{v}) = \hat{\rho}\vec{\nabla}\vec{v} = \hat{\rho}[\partial_x(-ay) + \partial_y(bx)] = \hat{\rho}(0+0) = 0$$

Stationary Euler equation:

$$(\vec{v}\vec{\nabla})\vec{v} = -\frac{1}{\hat{\rho}}\vec{\nabla}p + \vec{g}$$

With:

$$\begin{aligned} (\vec{v}\vec{\nabla})\vec{v} &= (v_x\partial_x v_x + v_y\partial_y v_x + v_z\partial_z v_x)\vec{e}_x \\ &+ (v_x\partial_x v_y + v_y\partial_y v_y + v_z\partial_z v_y)\vec{e}_y \\ &+ (v_x\partial_x v_z + v_y\partial_y v_z + v_z\partial_z v_z)\vec{e}_z \\ (\vec{v}\vec{\nabla})\vec{v} &= -abx\vec{e}_x - abye_y + 0e_z \end{aligned}$$

we obtain 3 equations for the 3 vector components of the Euler equation:

$$-abx = -\tfrac{1}{\hat{\rho}}\partial_x p \quad \Rightarrow p = \frac{1}{2}\hat{\rho}abx^2 + F(y,z) \tag{1}$$

$$-aby = -\tfrac{1}{\hat{\rho}}\partial_y p \quad \Rightarrow p = \frac{1}{2}\hat{\rho}aby^2 + G(x,z) \tag{2}$$

$$0 = -\tfrac{1}{\hat{\rho}}\partial_z p - g \quad \Rightarrow p = -\hat{\rho}gz + H(x,y) \tag{3}$$

which can be combined to:

$$p = \frac{1}{2}\hat{\rho}abx^2 + \frac{1}{2}\hat{\rho}aby^2 - \hat{\rho}gz + const.$$
$$p = \frac{1}{2}\hat{\rho}(abx^2 + aby^2 - 2gz) + p_0$$

Finally, using $p = \frac{\hat{\rho}}{m}kT$, we obtain:

$$T = \frac{1}{2}\frac{m}{k}(abx^2 + aby^2 - 2gz) + T_0$$

□

Problem 1.3:

1.3.1: Given:

$$\vec{v} = v_0 \cosh(x)\vec{e}_x + c\vec{e}_y$$

Equation for the streamlines $y(x)$:

$$\begin{aligned} \frac{dx}{v_x} &= \frac{dy}{v_y} \\ \frac{dy}{dx} &= \frac{v_y}{v_z} = \frac{c}{v_0 \cosh(x)} \\ y(x) &= y_0 + 2\frac{c}{v_0} \arctan(e^x) \end{aligned}$$

which can be plotted easily using a computer.

1.3.2:

$$\begin{aligned} \frac{d\vec{v}}{dt} &= \partial_t \vec{v} \quad +(\vec{v}\vec{\nabla})\vec{v} \\ \frac{d\vec{v}}{dt} &= 0 \quad +(v_x\partial_x v_x + v_y\partial_y v_x + v_z\partial_z v_x)\vec{e}_x \\ & \qquad +(v_x\partial_x v_y + v_y\partial_y v_y + v_z\partial_z v_y)\vec{e}_y \\ & \qquad +(v_x\partial_x v_z + v_y\partial_y v_z + v_z\partial_z v_z)\vec{e}_z \\ \frac{d\vec{v}}{dt} &= v_0^2 \cosh(x)\sinh(x)\vec{e}_x \end{aligned}$$

1.3.3: Stationary equation of continuity:

$$\begin{aligned} & \partial_t \rho + \vec{\nabla}(\rho\vec{v}) = 0 \\ \Rightarrow \quad & \rho\vec{\nabla}\vec{v} + \vec{v}\vec{\nabla}\rho = 0 \\ \Rightarrow \quad & \rho v_0 \sinh(x) + v_0 \cosh(x)\partial_x \rho + c\partial_y \rho = 0 \end{aligned}$$

Since the problem is invariant against shift in y, we assume that for the solution, $\partial_y \rho = 0$

$$\begin{aligned} 0 &= \rho v_0 \sinh(x) + v_0 \cosh(x) \partial_x \rho \\ \Rightarrow \quad \partial_x \rho &= -\rho \tanh(x) \\ \Rightarrow \quad \rho(x) &= e^{-\int \tanh(x') dx' + C} \\ \Rightarrow \quad \rho(x) &= \rho_0 e^{-ln(cosh(x))} \\ \Rightarrow \quad \rho(x) &= \frac{\rho_0}{cosh(x)} \end{aligned}$$