In 1D, "bosonization relations" of the following form hold:

\[ \psi^\dagger \psi - e^{-ip\phi} \]

fermion field \quad boson field

Klein factor

Goal of lectures:
- explain origin of these relations
- illustrate them with some canonical examples

Outline:
I. 1D-fermions, 1D-bosons
II. Bosonization identity
III. Impurity in Luttinger Liquid
IV. Kondo model

Popular applications
1. Interactions in 1D

Since fermions in 1D cannot pass each other, interactions are "strong" and dramatically change the physics (e.g. spin-charge separation)

Applications:
- nanotubes
- organic molecules
- semiconductor quantum wires
- quantum Hall edge states

Interactions in 1D:
\[ \int dx \, \gamma^i \gamma^j \gamma^k \gamma^l \sim \int dx (\partial_x \gamma_x)^2 \]

Kinetic energy:
\[ \int dx \, \gamma^i \gamma^i \sim \int dx (\partial_x \gamma_x)^2 \]

Interacting model becomes exactly solvable!
2. Impurity models (Kondo):
(Emery & Kivelson, '92)

Spin-flip term:
\[ S^+ \psi^+_{\downarrow}(\sigma) \psi_{\uparrow}(\sigma) + S^- \psi^+_{\uparrow}(\sigma) \psi_{\downarrow}(\sigma) \]

Bosonize:
\[ \psi_{\sigma} = e^{-i \phi_{\sigma}} \]

New boson field:
\[ \sim S^+ e^{-i \phi_s} + S^- e^{i \phi_s} \]

Refermionize:
\[ d^\dagger \psi_s + d \psi_s^{+} \quad \text{QUADRATIC}!! \]

Heuristic plausibility argument for bosonization relation

How can it be true that:
\[ \psi = e^{-i \phi} \quad ? \]  

(1)

For 1-D bosons, with linear dispersion:
\[ \langle \phi(x) \phi(x) \rangle \sim - \frac{1}{x} \]  

(2)

For 1-D fermions, with linear dispersion:
\[ \langle \psi(x) \psi^+(x) \rangle \sim - \frac{1}{x} \]  

(3)

or, using (1):
\[ \sim \langle e^{-i \phi(x)} e^{i \phi(x)} \rangle \]  

(4)

standard identity for bosonic operators:
\[ \sim e^{-\ln x} \sim \frac{i}{x} \]  

(5)

using (2):
\[ \sim e^{-\ln x} \sim \frac{i}{x} = (3) \]  

(6)
Questions:

How general is (5.1) ?

Does (5.1) rely on linear dispersion?  

Is (5.1) an operator identity?

Commutation relations?

Several species of electrons?

On what Fock space?

Role of cut-offs?

Finite-size effects?

Outline of lecture I: 1-D fermions & bosons

1. Linearization of fermion spectrum
2. Properties of 1d fermion fields
3. Role of cut-offs?
4. Normal ordering
5. Density fluctuations - bosonic excitations

I.1 Linearization of fermion spectrum (ignore spin)

$k = |p| - k_F$, or $p = \pm (k_F + k)$ for R/L-branches

Neglected terms [order $(k/k_F)$] describe curvature effects: current research topic.

Replacing (6.1) by (6.2) is justified if we are interested only in long-wavelength / low-energy properties, with \( |k| \ll \Gamma \) anyway, i.e. in excitation energies \( \omega, T, \nu \ll \varepsilon_0 \).

In this case, we may as well send cutoff \( \Gamma \to \infty \) and replace theory \( 1 \to 2 \).

Corresponding approximation for electron fields, step by step:

\[
\psi_{\text{phys}}(x) = \Delta_k \sum_p e^{ipx} c_p = \Delta_k \sum_k \left( e^{-i(k_0 + k)x} c_{k} + e^{i(k_0 + k)x} c_{-k} \right)
\]

\[\Delta_k = \left( \frac{2\pi}{L} \right)\]

Drop high-energy excitations, assuming they don’t matter for low-energy properties:

\[
\text{Step 1: drop } B \quad \Rightarrow \quad \psi_{\text{phys}}(x) = e^{-ik_0x} \psi_L(x) + e^{ik_0x} \psi_R(-x)
\]

with \( \psi_{L,R}(x) := \Delta_k \sum_{|k| < \Gamma} e^{-ikx} c_{k,L/R} \)

\[\text{I.2 Properties of 1d fermion fields}\]

Cutoff means: new fields \( \psi_{L/R}(x) \) can resolve spatial structures only if they are coarser than \( \Gamma \)

\[
\sum_{|k| < \Gamma} \quad \rightarrow \quad \lim_{\alpha \to 0} \sum_{k = -\infty}^{\infty} e^{-ik\alpha} \quad \rightarrow \quad \sum_{\Gamma \rightarrow a}
\]

Step 2: to get a mathematically simpler, cleaner theory, now take cutoff to infinity, i.e. add "positron states" (since they did not matter for low excitation energies anyway):

So, write: \( \eta = L, R \)

\[
\psi_{\eta}(x) = \Delta_k \sum_k e^{-ikx} c_k \eta
\]

Impose anti-periodic boundary conditions: (convenient to avoid degeneracy of Fermi ground state)

\[
\psi_L(-\xi) = -\psi_L(\xi) \quad \Rightarrow \quad k = \frac{2\pi}{\Delta}(n - \frac{1}{2})
\]
Anticommutators:
\[ \{ c_{k\eta}, c_{k'\eta'}^\dagger \} = 0, \quad \{ c_{k\eta}, c_{k'\eta'}^\dagger \} = \delta_{kk'} \delta_{\eta\eta'} \]

Continuum limit:
(finite bandwidth)
\[ \lim_{a \to 0} \sum_k \frac{\alpha k}{(k-x)^2 + \alpha^2} \]

Or:
infinite bandwidth
\[ k = \alpha (n - \eta) \quad n \in \mathbb{Z} \]

Linearized kinetic energy:
\[ H = \sum_{k\eta} \frac{\alpha k}{2} c_{k\eta}^\dagger c_{k\eta} \]

Fermi ground state:
\[ \begin{cases} k < 0 \text{ filled} : & c_{k\eta} |0\rangle = 0 \\ k > 0 \text{ empty} : & c_{k\eta} |0\rangle = 0 \end{cases} \]

Imaginary-time evolution:
\[ c_{k\eta}(\tau) := e^{-\frac{i\tau}{k}} c_{k\eta} e^{-\frac{\tau}{k}} \]
\[ = e^{-\frac{\tau}{k}} e^{i \frac{\tau}{k}} c_{k\eta} = e^{-\frac{k \tau}{2}} c_{k\eta} \]

If we ever need real-time evolution:
\[ c_k(t) = c_k(\tau \to -it) \]

Fermion field:
\[ \psi(x) = \Delta_{\zeta} \sum_k e^{-\frac{k}{\zeta}(x + \tau)} c_{k\eta} = \Delta_{\zeta} \sum_k e^{-\frac{k}{\zeta}} c_{k\eta} = \psi(\zeta) \]
I.3 Imaginary-time-ordered fermion correlator at $T = 0$

$$- S_{\eta \eta} (z) = \langle T \psi_{\eta}^{\dagger}(z) \psi_{\eta}(z) \rangle = \Theta(z) \langle \psi_{\eta}^{\dagger}(0) \psi_{\eta}(0) \rangle - \Theta(-z) \langle \psi_{\eta}^{\dagger}(w) \psi_{\eta}(w) \rangle$$

$$= \Delta \sum_{k k'} e^{-k z} \left[ \Theta(z) \langle \psi_{\eta}^{\dagger}(0) c_{k \eta} c_{k' \eta}(0) \rangle - \Theta(-z) \langle c_{k' \eta}^{\dagger} \psi_{\eta}(w) c_{k \eta} \psi_{\eta}(w) \rangle \right]$$

$$= \delta_{\eta \eta} \Delta \sum_{k > 0} e^{-k z} \sigma e^{-k a}$$

For finite $L$ one finds, using $k = \Delta L (n + i z)$, $y = e^{-\Delta L (\sigma z + a)}$:

$$- S_{\eta \eta} (z) = \Delta \sigma \sum_{n = 0} e^{-\sigma z i / L} = \Delta \sigma \frac{y^{-(\sigma + 1) / L}}{y^{\sigma / L} - y^{\sigma / L}} = \frac{\delta_{\eta \eta} e^{\pi (\sigma + 1) / L}}{\pi \sin \left[ \pi (z + \sigma a) \right]}$$

I.4 Fermion normal ordering

$$\{ \delta_{\eta \eta}, \text{ for } k > 0 \}$$

$$\{ \delta_{\eta \eta}, \text{ for } k < 0 \}$$

To bring "normal order" a product of operators, move all operators that annihilate the vacuum to the right of all others, and multiply by $(-1)$ for each exchange of two fermion operators.

For product of two operators, this is equivalent to:

$$x A B x^{\dagger} = A B - \langle 0 \mid A B \mid 0 \rangle$$

Example: $k > 0, k' < 0$:

$$x C_{k}^{\dagger} C_{k'} x^{\dagger} = - C_{k'}^{\dagger} C_{k}$$

$$= C_{k} C_{k'}^{\dagger} - \delta_{k k'}^{\dagger} \left( x C_{k}^{\dagger} C_{k'} x^{\dagger} \right)$$

By definition, vacuum expectation value of two normal ordered operators:

$$\langle 0 \mid x A B x^{\dagger} \mid 0 \rangle = 0$$
I.4 Fermion normal ordering

To bring "normal order" a product of operators, subtract their vacuum expectation value:

\[ \langle \hat{\mathcal{X}} \cdot \hat{\mathcal{A}} \hat{\mathcal{B}} \hat{\mathcal{C}} \hat{\mathcal{D}} \cdots \rangle = \hat{\mathcal{X}} \hat{\mathcal{Y}} \hat{\mathcal{Z}} \hat{\mathcal{W}} \cdots \langle 0 \rangle - \langle \mathcal{X} \mathcal{Y} \mathcal{Z} \mathcal{W} \cdots \rangle \]

Equivalently: move all operators that annihilate the vacuum to the right of all others, and multiply by (-1) for each exchange of two fermion operators.

For example:

\[ C_{k'} C_k \]

By definition, vacuum expectation value of

\[ \langle \hat{\mathcal{X}} \cdot \hat{\mathcal{A}} \hat{\mathcal{B}} \hat{\mathcal{C}} \hat{\mathcal{D}} \cdots \rangle = 0 \]

I.5 Density fluctuations - bosonic excitations

(2 pi) density:

\[ \rho(x) = \frac{1}{(2\pi)^3} \bar{\psi}(x) \bar{\psi}(x) \]

Fourier representation:

\[ \rho(q) = \sum_{k} e^{i(k \cdot x)} \bar{c}_k c_k \]

where we defined:

Particle number relative to Fermi ground state:

\[ \hat{N} = \sum_k \bar{c}_k c_k \]

Momentum lowering op:

\[ b_{k} = \frac{i}{\sqrt{N}} \sum_k c_{k+q} c_k \]

Momentum raising op:

\[ b_{k}^\dagger = \frac{i}{\sqrt{N}} \sum_k c_{k-q} c_k \]

Note: (5) and (6) are automatically normal ordered, hence no need to write
Bosonic commutation relations: (for notational simplicity, below we drop the index $\eta$)

$$\left[ N, b_{q}^{\dagger} \right] = \frac{i}{n_{k}^{2}} \sum_{k, k'} \left[ c_{k}^{\dagger} c_{k'}, c_{k'-q}^{\dagger} c_{k'} \right]$$

$$= \frac{i}{\sqrt{n_{k}^{2}}} \sum_{k, k'} \left( c_{k}^{\dagger} c_{k'} \delta_{k, k'-q} - c_{k'-q}^{\dagger} c_{k} \delta_{k, k'} \right)$$

$$= \frac{i}{\sqrt{n_{k}^{2}}} \sum_{k} \left( c_{k}^{\dagger} c_{k+q} - c_{k-q}^{\dagger} c_{k} \right) = 0$$

Similarly:

$$\left[ b_{q}, b_{q'}^{\dagger} \right] = 0, \quad \left[ b_{q}, b_{q'} \right] = 0, \quad \left[ b_{q}^{\dagger}, b_{q'}^{\dagger} \right] = 0$$

$$\left[ b_{q}, b_{q'}^{\dagger} \right] = 0, \quad \left[ b_{q}, b_{q'} \right] = 0, \quad \left[ b_{q}^{\dagger}, b_{q'}^{\dagger} \right] = 0$$

$$S_{kk'}^{t} = \left[ b_{k}^{\dagger}, b_{k'}^{\dagger} \right] = \frac{1}{n_{k}^{2}} \sum_{k} \left[ c_{k-1}^{\dagger} c_{k}, c_{k+1}^{\dagger} c_{k'} \right]$$

$$= \frac{1}{n_{k}^{2}} \sum_{k, k'} \left[ c_{k-1}^{\dagger} c_{k'}, c_{k'-1}^{\dagger} c_{k} - c_{k+1}^{\dagger} c_{k} \delta_{k-1, k'} \delta_{k, k'} \right]$$

$$= \frac{1}{n_{k}^{2}} \sum_{k} \left[ c_{k-1}^{\dagger} c_{k-1}, c_{k+1}^{\dagger} c_{k} \right]$$

if $q + q'$, both terms are normal-ordered, so we can set $k + q' \rightarrow k$ here, obtaining $0$

if $q = q'$, both terms have to be normal-ordered first, before rearranging sum; this gives:

number of possible transitions generated by

$$\delta_{q_{1}q_{2}} \times \frac{1}{n_{k}^{2}} \sum_{k} \left[ c_{k-1}^{\dagger} c_{k}, c_{k+1}^{\dagger} c_{k} \right]$$
6. Properties of 1d Boson fields

"annihilation field":
\[ \tilde{\eta}_{\eta}(x) = - \sum_{\eta > 0} e^{-aq/2} \frac{1}{\sqrt{n_{\eta}}} e^{-i q \cdot x} b_{\eta} \]

"creation field":
\[ \tilde{\eta}^\dagger_{\eta}(x) = - \sum_{\eta > 0} e^{-aq/2} \frac{1}{\sqrt{n_{\eta}}} e^{i q \cdot x} b_{\eta}^\dagger \]

The ultraviolet cutoff \( a \) here acts as a bandwidth for bosonic excitations. In fermion language, it sets the maximum momentum difference between particle/hole pairs.

Hermitian boson field:
\[ \phi_{\eta}(x) = \tilde{\eta}_{\eta}(x) + \tilde{\eta}^\dagger_{\eta}(x) = \phi_{\eta}^D(x) \]

Derivative gives density:
\[ \partial_x \phi_{\eta}(x) = \Delta_c \sum_{\eta \neq 0} i \sqrt{n_{\eta}} \left( e^{-i q \cdot x} b_{\eta} - e^{i q \cdot x} b_{\eta}^\dagger \right) \]

(provided \( a = 0 \))

Compare (16.4) & (13.3):
\[ \rho(x) = (\eta_{\eta}(x) \eta_{\eta}(x))^* = \Delta_c n_{\eta}^2 + \partial_x \phi_{\eta}(x) \]

Boson field commutators:
\[ [b_{\eta}, b_{\eta}^\dagger] = [b_{\eta}^\dagger, b_{\eta}^\dagger] = 0 \Rightarrow [\phi(x), \phi(x')] = 0 , \quad [\phi(x), \phi'(x')] = 0 \]

(for notational simplicity, below we drop the index \( \eta \) )

\[ [b_{\eta}, b_{\eta}^\dagger] = \delta_{\eta \eta'} \Rightarrow [\phi(x), \phi'(x')] = \sum_{\eta \neq 0} \frac{i}{\sqrt{n_{\eta}}} e^{-i q \cdot x} e^{i q' \cdot x'} [b_{\eta}, b_{\eta}^\dagger] \]

\[ \phi = \Delta_c n \]
\[ y = e^{-i \Delta_c (x-x' - i a)} \]
\[ \Delta_c \rightarrow 1 - i \Delta_c \]

\[ \Delta_c = \frac{\xi}{c} \]

\[ \frac{\partial_{\phi}}{\Delta_c \rightarrow 0} = \ln ( i \Delta_c (x-x' - i a) ) \]

\[ \ln ( \Delta_c ) \]

Note: this commutator needs both infrared and ultraviolet regulators, \( 1/L \) and \( a \).
Commutator of $\phi$ with its derivative

$$[\phi(x), \partial_x \phi(x')] = \left[ \phi(x), \partial_x \phi(x') \right] + \left[ \phi(x), \partial_x \phi(x') \right]$$  \hspace{1cm} (1)

$$[\phi(x), \phi(x')] = \frac{i a}{2} \left( \frac{a}{(x-x')^2 + a^2} - \frac{\pi}{L} \right)$$  \hspace{1cm} (2)

$$\lim_{L \to \infty} \int \frac{dx' z}{(x-x')^2 + a^2} = \frac{2\pi i}{L} \delta_a(x-x)$$  \hspace{1cm} (3)

The $1/L$ term ensures consistency upon integrating (1):

$$\int_{-L/2}^{L/2} dx' \left[ \phi(x), \partial_x \phi(x') \right] = \frac{2\pi i}{L} \left[ \delta_a(x) - \frac{1}{L} \right] = 0$$  \hspace{1cm} (4)

Commutator of $\phi$ with itself

$$[\phi(x), \phi(x')] = \int_{-L/2}^{L/2} dx' \left[ \phi(x), \partial_x \phi(x') \right] + c$$  \hspace{1cm} (5)

fixed by requiring commutator to vanish for $x = x'$.

$$\int_{-L/2}^{L/2} dx' \left[ \phi(x), \partial_x \phi(x') \right] = \frac{2\pi i}{L} \left[ 1 - 1 \right] = 0$$  \hspace{1cm} (6)

$$\int_{-L/2}^{L/2} dx' \left[ \phi(x), \phi(x') \right] = \frac{2\pi i}{L} \left[ 1 - 1 \right] = 0$$  \hspace{1cm} (7)

$$\phi(x) \quad \text{consistent}$$  \hspace{1cm} (8)

since $\phi$ is periodic, (16.1, 16.2)

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