Lecture III - Odds and Ends

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Original fermion field
(with finite bandwidth for k,
i.e. for energy of particles or holes)

$$\psi_\eta(x) = \Delta^{1/2} \sum_k e^{-|k|a} e^{-ikx} c_{k\eta} , \qquad (1)$$

Boson field
(with finite bandwidth for q,
i.e. for energy of particle-excitations)

$$\varphi_\eta(x) = -\sum_{q>0} e^{-qa/2} \frac{1}{\sqrt{n_q}} e^{-iqx} \overbrace{b_{q\eta}}^{-\sum_k c^\dagger_{k-q} c_k} \qquad (2)$$

Define new fermion field
(with finite bandwidth for q, via phi):

$$\boxed{\psi^{(a)}_\eta(x) := F_\eta \underbrace{\hat\lambda_\eta(x)}_{\Delta_L^{1/2} e^{-i(N_\eta - 1/2)x}} e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)}} \qquad (3)$$

Comments:

1. Eq. (3) does not require a cutoff (we may set a = 0), because exponentials on RHS are normal ordered:
$${}_0\langle \vec N | \underbrace{e^{-i\varphi^\dagger_\eta(x)}}_{=1} \underbrace{e^{-i\varphi_\eta(x)}}_{=1} | \vec N \rangle_0 = 1 \qquad b|\vec N\rangle_0 = 1$$
2. For $a = 0$ we have an operator identity between new and old fields: $\psi^{(0)}_\eta(x) = \psi_\eta(x)$
3. For $a \neq 0$ the new field $\psi^{(a)}_\eta(x)$ is not identically equal to old field: $\psi^{(a)}_\eta(x) \neq \psi_\eta(x)$
Their long-distance behavior is the same (this is what we are interested in),
but short-distance behavior on scale of a is different (we don't care about it anyway).
4. Advantage of $a \neq 0$: two exponentials factors can be combined (unnormal-ordered).

Unnormal-ordering the bosonic exponentials

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$$\psi^{(a)}_\eta(x) \overset{(1.3)}{:=} F_\eta \hat\lambda_\eta(x) \underbrace{e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)}} \qquad (1)$$

(II.10.2iii) $e^A e^B = e^{A+B+\frac{1}{2}\underbrace{[A,B]}_{c}}$ if $[A,c] = [B,c] = 0$

$$= F_\eta \cancel{\Delta_L^{1/2}} e^{-i\Delta_L(N_\eta - \frac{1}{2})x} e^{-i(\underbrace{\varphi^\dagger_\eta + \varphi_\eta}_{\phi_\eta(x)})} e^{-\frac{1}{2}\underbrace{[\varphi^\dagger_\eta(x), \varphi_\eta(x)]}_{\ln \cancel{\Delta_L} a}} \qquad (2)$$

(divergent prefactor arises because of unnormal-ordering)

Various common notations:

$$\psi^{(a)}_\eta(x) = a^{-1/2} F_\eta(x) e^{-i\phi_\eta(x)}, \qquad F_\eta(x) := F_\eta e^{-i\Delta_L(N_\eta - \frac{1}{2})x} \qquad (3)$$

$$\psi^{(a)}_\eta(x) = a^{-\frac{1}{2}} F_\eta e^{-i\Phi_\eta(x)}, \qquad \Phi_\eta(x) := \phi_\eta(x) + \Delta_L(N_\eta - 1/2)x \qquad (4)$$

"zero mode"

$$\psi^{(a)}_\eta(x) = a^{-\frac{1}{2}} e^{-i\tilde\Phi_\eta(x)}, \qquad \tilde\Phi_\eta(x) := \Phi_\eta(x) - \theta_\eta, \qquad F_\eta := e^{-i\theta_\eta} \qquad (5)$$

Popular (but "dangerous") notation (I do not recommend using it, but you should know about it)

Introduce "phase operators" conjugate to N:

$$F^\dagger_\eta = e^{i\theta_\eta}, \quad F_\eta = e^{-i\theta_\eta}, \tag{1}$$

$$[\hat N_\eta, i\theta_{\eta'}] := \delta_{\eta\eta'} \quad \text{"sloppy"} \qquad [\hat N_\eta, e^{\pm i\theta_{\eta'}}] = \pm\delta_{\eta\eta'} e^{\pm i\theta_{\eta'}} = (7.6), (7.7) \tag{2}$$

(theta's are sometime called

$$[\theta_\eta, \theta_{\eta'}] := \left\{\begin{matrix} i\pi \\ 0 \\ -i\pi \end{matrix}\right\} \text{ if } \eta \left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} \eta' \tag{3}$$

(1), (2), (3) reproduce (II.7.4) to (II.7.7), using

$$\left.\begin{aligned} [A, e^B] &= c\, e^B \\ e^A e^B &= e^B e^A e^c \end{aligned}\right\} \text{ if } c = [A,B] = c\text{-number} \tag{4}$$

But, (2) is sloppy notation that produces a contradiction:

$$0 = (N_\eta - N_\eta)\langle N_\eta|\theta|N_\eta\rangle \neq \underbrace{\langle N_\eta|\hat N_\eta}_{\neq \langle N|N_\eta} i\theta_\eta - i\theta_\eta \hat N_\eta|N_\eta\rangle \overset{(2)}{=} \langle N_\eta|1|N_\eta\rangle = 1 \tag{5}$$

What went wrong?

Theorem: If X and Y are conjugate operators, meaning [X, iY] = 1, and the spectrum of X is the set of discrete integers, then X is hermitian only in the space of states produced by acting on a reference state by periodic functions of Y, in other words, functions of exp(iY) .

But: (11) contains states not periodic in theta, namely $\hat\theta|N_\eta\rangle$ so $\hat N$ is not hermitian!

Commutators of bosonized expressions (use (1.3) with a = 0)

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} \sim \{F_\eta, F^\dagger_{\eta'}\} = 0 \quad \text{if } \eta \neq \eta' \tag{1}$$

(suppress eta-index henceforth)

$$FN = (N+1)F \quad (\text{II}.7.6)$$

$$\eta = \eta': \; = \Big(\Delta_L^{1/2} \cancel{F}\, e^{-i\Delta_L(N-\frac12)x} e^{-i\varphi^\dagger(x)} e^{-i\varphi(x)} e^{i\varphi^\dagger(x')} e^{i\varphi(x')} e^{i\Delta_L(N-\frac12)x'} \underbrace{F F^\dagger}_{=1} \Delta_L^{1/2}$$

$$+ \Delta_L^{1/2} e^{i\varphi^\dagger(x')} e^{i\varphi(x')} e^{i\Delta_L(N-\frac12)x'} \underbrace{F^\dagger F}_{=1} e^{-i\Delta_L(N-\frac12)x} e^{-i\varphi^\dagger(x)} e^{-i\varphi(x)} \Delta_L^{1/2} \Big) \tag{2}$$

$$= \Big[\Delta_L \underbrace{e^{-i\Delta_L(N-\frac12)(x-x')}}_{=1 \;\; (e^{-i\frac{2\pi}{2}\frac12\bar n L}) = (-1)^{\bar n}} \cdot \underbrace{e^{-i(\varphi^\dagger(x) - \varphi^\dagger(x'))} e^{-i(\varphi(x) - \varphi(x'))}}_{=1, \text{ since } \varphi(x) = \varphi(x+\bar n L)}\Big] A(x,x') \tag{3}$$

if $x - x' \overset{(5)}{=} \bar n L$:

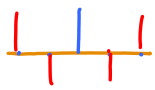
$$A(x,x') = \Big(e^{[\varphi(x), \varphi^\dagger(x')]} \overbrace{e^{-i\Delta_L(x-x')}}^{y} + e^{[\varphi(x'), \varphi^\dagger(x)]}\Big) \tag{4}$$

$$= \left(\frac{y}{1-y} + \frac{1}{1-y^{-1}}\right) = \sum_{n=0}^{\infty}\left(y^{n+1} + y^{-n}\right) = \sum_{n\in\mathbb{Z}} y^n = L\sum_{\bar n} \delta(x - x' - \bar n L) \tag{5}$$

$$e^A e^B \overset{(2.10.2iii)}{=} e^B e^A e^{[A,B]}$$

$$[\varphi(x), \varphi^\dagger(x')] \overset{(\text{I}.17.4),\, a=0}{=} -\ln(1-y), \quad y = e^{-i\Delta_L(x-x')}$$

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} \overset{(3)}{=} 2\pi \sum_{\bar n} \delta(x - x' - \bar n L)(-1)^{\bar n} \quad \checkmark$$



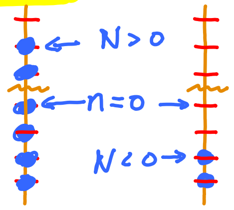
antiperiodic

Bosonizing linearized kinetic Hamiltonian

(suppress index η below) $v_F = 1$ **5**

Linearized fermionic kinetic Hamiltonian:
$$\boxed{H_0 = \sum_{k=-\infty}^{\infty} k :c^\dagger_{k\eta} c_{k\eta}: , \quad k = \Delta_L (n - \tfrac{1}{2})} \quad (1)$$

Energy of N-particle ground state $|N\rangle_0$:
$$E_0^N = {}_0\langle N|H|N\rangle_0 = \Delta_L \begin{cases} \sum_{n=1}^{N} (n - \tfrac{1}{2}) & \text{for } N \geq 0 \\ \sum_{n=N+1}^{0} -(n-\tfrac{1}{2}) & \text{for } N < 0 \end{cases} = \frac{\Delta_L}{2} N^2 \xrightarrow{L\to\infty} 0 \quad (2)$$



Consider:
$$[H_0, b^\dagger_q] = \sum_{kk'} k \, [c^\dagger_k c_k, c^\dagger_{k'+q} c_{k'}] \frac{i}{\sqrt{n_q}} = \sum_k k \left(c^\dagger_k c_{k-q} - c^\dagger_{k+q} c_k \right) \frac{i}{\sqrt{n_q}} = q \, b^\dagger_q \quad (3)$$

$\delta_{kk'}$ $\quad \delta_{k,k'+q}$ $\quad k+q$ $\quad k \to k+q$

Thus, boson creation op. are energy ladder op:
$$H_0 \, b^\dagger_q |N\rangle_0 = (E_0^N + q) \, b^\dagger_q |N\rangle_0 \quad (4)$$

The only bosonic operator that also satisfies (2) and (3) for all q is:
$[H_0, b^\dagger_q] = q \, b^\dagger_q$
$$\boxed{H_0 := \sum_{q>0} q \, b^\dagger_q b_q + \frac{\Delta_L}{2} \hat{N}^2} \quad (5)$$

(seemingly quartic in $c^\dagger c c^\dagger c$!!)

hence: (1) = (5)

I. Imaginary-time-ordered boson correlator at T = 0

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$b_q(\tau) = e^{-\tau q} b_q$

Imaginary-time evolution:
$$\phi(\tau, x) \overset{(\mathrm{II}.2.9)}{=} - \sum_{q>0} \frac{e^{-aq/2}}{\sqrt{n_q}} \left[e^{-q\underbrace{(ix+\tau)}_{z}} b_q + e^{q(ix+\tau)} b^\dagger_q \right] = \phi(z) \quad (1)$$

$$\boxed{{}_0\langle 0| \mathcal{T} \phi(z) \phi(0) |0\rangle_0}$$

$\langle \phi(-z)\phi(0) \rangle$ by time translational invariance
$$= \theta(\tau) \langle \phi(z) \phi(0) \rangle + \theta(-\tau) \langle \phi(0) \phi(z) \rangle'' \quad (2)$$

$\sigma = \mathrm{sign}(\tau)$
$$= \langle \phi(\sigma z) \phi(0) \rangle , \quad \text{in } \langle (b_q + b^\dagger_q)(b_{q'} + b^\dagger_{q'}) \rangle \text{ only } b_q b^\dagger_q \delta_{qq'} \text{ contributes} \quad (3)$$

$$= \sum_{q>0} \frac{e^{-aq}}{n} e^{-qz\sigma} \underbrace{\langle b_q b^\dagger_q \rangle}_{=1} \quad \text{with } q = \Delta_L n , \quad y = e^{-\Delta_L (z\sigma + a)} \quad (4)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} y^n = -\ln(1-y) = -\ln\left(1 - e^{-\Delta_L(z\sigma + a)}\right) \xrightarrow{L\to\infty} \boxed{-\ln\left(\Delta_L(\sigma z + a)\right)} \quad (5)$$

Time evolution of Klein factor:
$$F_\eta(\tau, x) = e^{H\tau} F_\eta \, e^{\mp i \Delta_L (N_\eta - \frac{1}{2}) x} e^{-H\tau} = F_\eta \, e^{\mp \Delta_L \underbrace{(N_\eta - \frac{1}{2})(ix+\tau)}_{\xrightarrow{L\to\infty} 1}} \quad (6)$$

$F^\dagger_\eta \quad F^\dagger_\eta \quad F^\dagger_\eta$

Using bosonization to calculate fermion correlators

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Theorem: for free boson Hamiltonian $H = \sum_q \omega_q b_q^\dagger b_q$ and $\hat B = \sum_q (\lambda_q b_q + \bar\lambda_q b_q^\dagger)$

(ground state or) thermal expectation values of exponentials of bosons satisfy: (App C vDS)

$$\langle e^{\hat B}\rangle = e^{\frac12\langle \hat B^2\rangle}, \quad \text{where } \langle \hat O\rangle = \mathrm{Tr}(e^{-\beta H}\hat O)/\mathrm{Tr}\, e^{-\beta H} \qquad (1)$$

(1) and (II.10.2iii) imply:

$$\langle e^{\hat B_1} e^{\hat B_2}\rangle = e^{\langle \hat B_1 \hat B_2 + \frac12 \hat B_1^2 + \hat B_2^2\rangle} \qquad (2)$$

Bosonize fermion correlator:

$$\boxed{\langle 0|T\psi(z)\,\psi^\dagger(0)|0\rangle} = \frac1a \langle 0|T\, \underbrace{F F^\dagger}_{=1} e^{-\Delta_L(\underset{N=1}{N}-1/2)z} e^{-i\phi(z)} e^{i\phi(0)} \cancel{F^\dagger}|0\rangle \qquad (3)$$

$$\overset{(2)}{=} \frac{\sigma}{a} e^{-\Delta_L z/2} e^{\langle 0|T\phi(z)\phi(0) - \frac12\phi(z)\phi(z) - \frac12\phi(0)\phi(0)|0\rangle} \qquad (4)$$

$$\xrightarrow{L\to\infty} \frac{\sigma}{a} e^{-(\ln \cancel{\Delta_L}(\sigma z+a) - 2\cdot\frac12 \ln \cancel{\Delta_L} a)}$$

$$\langle T\phi(z)\phi(0)\rangle \overset{(6.5)}{=} -\ln\Delta_L(\sigma z+a)$$

$$= \frac{\cancel{\sigma}}{\cancel{a}}\,\frac{\cancel{a}}{\cancel{\sigma}z + a\sigma} = \boxed{\frac{1}{z+\sigma a}} = (\mathrm{I}.11.5)$$

[if L is kept finite, one recovers (I.11.6)]

Vertex operators: general exponentials of boson fields

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Definition of "vertex operator": with charge $\lambda \in \mathbb{R}$

$$V_\lambda^{(\eta)}(z) := \Delta_L^{\lambda^2/2}\ \overset{\text{see (2.2)}}{{}^{\times}_{\times} e^{i\lambda\phi_\eta(z)} {}^{\times}_{\times}} = a^{-\lambda^2/2} e^{i\lambda\phi_\eta(z)} \qquad (1)$$

$$:= e^{i\lambda\varphi_\eta^\dagger(z)} e^{i\lambda\varphi_\eta(z)}$$

unnormal-order, producing a factor $e^{-\lambda^2[\varphi^\dagger(z),\varphi(z)]} = e^{-\lambda^2 \ln \Delta_L a}$

Ground state expectation value:

$$\langle\vec 0|V_\lambda^{(\eta)}(z)|\vec 0\rangle_0 = \Delta_L^{\lambda^2/2} \xrightarrow[\Delta_L\to0]{L\to\infty} \delta_{\lambda 0} \qquad (2)$$

Two-point correlator:

$$\langle\vec 0|T\, V_\lambda^{(\eta)}(z) V_{\lambda'}^{(\eta')}(z')|\vec 0\rangle_0 = \delta_{\eta\eta'}\, \cancel{a^{-(\lambda^2+\lambda'^2)/2}}\, e^{-\lambda\lambda'\underbrace{\langle T\phi(z)\phi(0)\rangle}_{-\ln\Delta_L(\sigma z+a)} - \frac12(\lambda^2+\lambda'^2)\underbrace{\langle\phi(0)\phi(0)\rangle}_{-\ln\Delta_L \cancel{a}}}$$

$$0 \neq \langle e^{+i\lambda\phi} e^{-i\lambda\phi}\rangle \qquad = \delta_{\eta\eta'}\frac{\Delta_L^{\frac12(\lambda+\lambda')^2}}{(\sigma z+a)^{-\lambda\lambda'}} \xrightarrow[\Delta_L\to0]{L\to\infty} \delta_{\eta\eta'}\frac{\delta_{\lambda,-\lambda'}}{(\sigma z+a)^{\lambda^2}} \qquad (3)$$

Similarly for n-point correlator:

$$\langle\vec 0|T\, V_{\lambda_1}^{(\eta)}(z_1)\dots V_{\lambda_n}^{(\eta')}(z_n)|\vec 0\rangle_0 = \Delta_L^{\frac12\left(\sum_{j=1}^n \lambda_j\right)^2} \prod_{i<j} (z_{ij}\sigma_{ij}+a)^{\lambda_i\lambda_j} \qquad (4)$$

"charge neutrality" $\xrightarrow[\Delta_L\to0]{L\to\infty} = 0$ unless $\sum_{j=1}^n \lambda_j = 0$

[$z_{ij} = z_i - z_j$, $\sigma_{ij} = \mathrm{sign}(\tau_i - \tau_j)$]

Kinetic energy in position space (spinless electrons, L / R) $\eta = L, R$

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fermionic description: $$\psi_{L/R}(x) \overset{(t+x)\,(\mathrm{I}.8.1)}{=} \Delta_L^{1/2} \sum_k e^{-ikx} c_{kL,R} =: \tilde{\psi}_{L/R}(\pm x) \qquad (1)$$

(mathematical L-movers) (mathematical L / R -movers)

Kinetic energy: $$H_0 \overset{(5.1)}{=} \sum_{\eta=L,R} \sum_{k=-\infty}^{\infty} k :c^\dagger_{k\eta} c_{k\eta}: \qquad (2)$$

$$= \int_{-L/2}^{L/2} \frac{dx}{2\pi} :\Big[\tilde{\psi}_L^\dagger(x)\,(i\partial_x)\,\tilde{\psi}_L(x) + \tilde{\psi}_R^\dagger(x)\,(-i\partial_x)\,\tilde{\psi}_R(x)\Big]: \qquad (3)$$

$\hookrightarrow k$ $\hookrightarrow k$

bosonic description: $$\phi_{L/R}(x) \overset{(\mathrm{II}.2.9)}{=} -\sum_{q>0} \frac{e^{-aq/2}}{\sqrt{n_q}} \Big[e^{-iqx} b_{qL,R} + e^{iqx} b^\dagger_{qL,R}\Big] =: \tilde{\phi}_{L/R}(\pm x) \qquad (4)$$

(mathematical L-movers) (mathematical L / R -movers)

Kinetic energy: $$H_0 \overset{(5.5)}{=} \sum_{\eta=L,R} \Big[\sum_{q>0} q\, b^\dagger_{q\eta} b_{q\eta} + \frac{\Delta_L}{2} \hat{N}_\eta^2\Big] \qquad (5)$$

$$= \sum_{\eta=L,R} \Big[\int_{-L/2}^{L/2} \frac{dx}{2\pi} :\tfrac{1}{2}(\partial_x \tilde{\phi}_\eta)^2: + \frac{\Delta_L}{2} \hat{N}_\eta^2\Big] \qquad (6)$$

$\phi \sim b + b^\dagger$

$c^\dagger c \quad c^\dagger c$

$\left(\frac{1}{\sqrt{n_q}} n_q\right)^2 \sim q$

Electron-electron interactions

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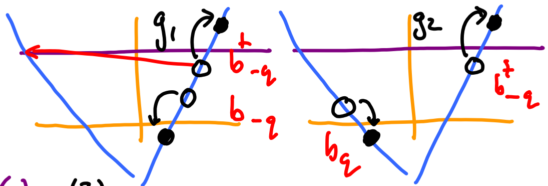
$$\hat{V}_{ee} = \frac{1}{2L} \sum^{\text{all}}_{kk'q} V_{ee}(q)\, c^\dagger_{k-q} c^\dagger_{k'+q} c_{k'} c_k \qquad (1)$$



$$= \frac{1}{2} \int dx\, dx'\; \psi^\dagger(x) \psi^\dagger(x') V_{ee}(x-x') \psi(x') \psi(x) \qquad (2)$$

$\rho_R(x)$ $\rho_R(x')$ $\delta(x-x')$

$$\psi_{phys} = e^{-ik_Fx} \tilde{\psi}_L(x) + e^{ik_Fx} \tilde{\psi}_R(x) \qquad (3)$$

$$= \frac{1}{2} \int \frac{dx}{2\pi} \frac{dx'}{2\pi} \delta(x-x') \Big\{ \frac{g_4}{2} \big[\tilde{\psi}^\dagger_R(x) \tilde{\psi}^\dagger_R(x') \tilde{\psi}_R(x') \tilde{\psi}_R(x)\big] + R \leftrightarrow L$$

$$g_2 \big[R^\dagger \; L^\dagger \; L \; R\big] \quad "$$

$$e^{-2ik_Fx} \big[R^\dagger \; R \; R^\dagger \; L\big] \quad " \qquad (4)$$

$$e^{-4ik_Fx} \big[R^\dagger \; R^\dagger \; L \; L\big] \quad "$$

$e^{i\phi_L}$ $e^{i\phi_R}$

Umklapp $q \sim 2k_F$

double-Umklapp $q \sim 4k_F$

drop: high-energy!

Tomonaga-Luttinger model:

$$H_{int} = \int_{-L/2}^{L/2} \frac{dx}{2\pi} :\Big[g_2\, \tilde{\rho}_L(x) \tilde{\rho}_R(x) + \frac{1}{2} g_4 \big(\tilde{\rho}_L^2(x) + \tilde{\rho}_R^2(x)\big)\Big]: \qquad (5)$$

this is already in bosonized form, since

$$\tilde{\rho}_{L/R}(x) \overset{(\mathrm{II}.2.10)}{=} \pm \partial_x \tilde{\phi}_{L/R}(x) + \Delta_L \hat{N}_{L/R} \qquad (6)$$