Functional Field Integral (Altland & Simons, Chapter 4)

04.11.08 $\boxed{\text{FFI 1}}$

Comment: if you have time, read as much of chapter 3 as you can! (I skip it, because it deals with single-particle quantum mechanics, hence would belong in a QM2 course).

Single-particle path integral:

$$\langle q_f | e^{-i\hat{H}t/\hbar} | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} Dx \; e^{\frac{i}{\hbar}\underbrace{\int_0^t dt' \left(p\dot{q} - H(p,q)\right)}_{S[q(t)]}} \quad (1)$$

Goal here: calculate many-body partition functions / correlation functions, e.g.

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \quad (2)$$

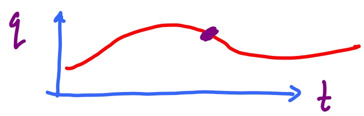
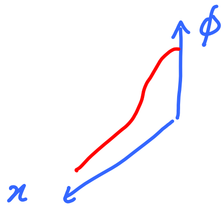
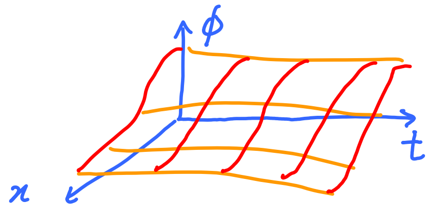
many-body eigenstates of $\hat{H}$

using field integral representation:

$$Z = \int D(\bar{\psi},\psi)\, e^{-S[\bar{\psi},\psi]}, \quad S[\bar{\psi},\psi] = \int_0^\beta d\tau \left[\bar{\psi}\partial_\tau\psi + H(\bar{\psi},\psi) - \mu N(\bar{\psi},\psi)\right] \quad (3)$$

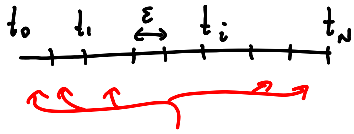
bosonic or fermionic coherent states — imaginary time

$\boxed{\text{FF2}}$

	Degrees of freedom	Path integral
single-particle QM	q	
many-particle quantum field theory		

$$Z = \sum_n \langle n | \underbrace{e^{-\frac{\hat{K}\varepsilon}{N}} \cdots\cdots e^{-\frac{\hat{K}\varepsilon}{N}}}_{N \text{ factors}} | n \rangle \quad (4)$$

$\beta\hat{H} - \mu\hat{N} = \hat{K}$



insert $1 = \sum_m |m\rangle\langle m|$

Eigenstates of $\hat{H}$ in general not known.

Since many-body $H = H(a, a^\dagger)$, it will be convenient to resolve $1 = \sum |\;\rangle\langle\;|$ in terms of eigenstates of a

Boson Coherent States

Let $\{a_i\}$ be set of boson operators, $[a_i, a_j^\dagger] = \delta_{ij}$ (5)

We seek eigenstate of $\{a_i\}$ such that

$$a_i|\phi\rangle = \phi_i|\phi\rangle \quad , \text{ where } \phi = \{\phi_i\} \quad , \phi_i \in \mathbb{C} \tag{6}$$

General form:
$$|\phi\rangle = \sum_{n_1, n_2 \dots} C_{n_1, n_2, \dots} \underbrace{|n_1, n_2, \dots\rangle}_{\equiv \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \cdots |0\rangle} \tag{7}$$

Bosonic vacuum

Claim: solution of (6):

$$|\phi\rangle = \exp\left[\sum_j \phi_j a_j^\dagger\right]|0\rangle \tag{8}$$

if $[A,B]$ = c-number
$e^{-B} A e^{B} = A + [A,B]$
$A e^{B} = e^{B}(A + [A,B])$

Proof: $a_i|\phi\rangle = \exp\left[\sum_j \phi_j a_j^\dagger\right]\left(a_i + \sum_j \underbrace{[a_i, \phi_j a_j^\dagger]}_{\delta_{ij}\phi_i}\right)|0\rangle$

$= \phi_i |\phi\rangle$ ✓ (9)

Properties:

$(6)^\dagger$: $\langle\phi|a_i^\dagger = \langle\phi|\bar{\phi}_i$, (10)

where $\bar{\phi}_i$ = c.c. of ϕ_i , $\langle\phi| = \langle 0|\exp\left[\sum \bar{\phi}_\cdot a_\cdot\right]$ (11)

$$\boxed{\partial_{\phi_i}|\phi\rangle} = \partial_{\phi_i} \sum_{n=-1}^{\infty} \frac{1}{(n+1)!}\left(\sum_j \phi_j a_j^\dagger\right)^{n+1}|0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{a_i^\dagger}{n!}\left(\sum_j \phi_j a_j^\dagger\right)^{n}|0\rangle \boxed{= a_i^\dagger|\phi\rangle} \tag{13}$$

$(13)^\dagger$ $\langle\phi|a_i = \partial_{\bar{\phi}_i}\langle\phi|$ (13')

Check:

$$[a_i, a_j^\dagger]|0\rangle = \delta_{ij}|0\rangle = \left(a_i\partial_{\phi_j} - a_i^\dagger\phi_i\right)|0\rangle = \underbrace{(\partial_{\phi_i}\phi_i - \phi_i\partial_{\phi_i})}_{\delta_{ij}}|0\rangle \tag{14}$$
✓

Scalar product:

$$\boxed{\langle\theta|\phi\rangle} = \langle 0|e^{\sum_i \bar{\theta}_i a_i} e^{\sum_j \phi_j a_j^\dagger}|0\rangle \boxed{= e^{\sum_i \bar{\theta}_i \phi_i}} \tag{15}$$

$e^A e^B =$
$e^B e^A e^{[A,B]}$

Norm: $\langle\phi|\phi\rangle = \exp\sum_j|\phi_j|^2$ (16) FFI 5

Completeness: (Over- ") since $\langle\theta|\phi\rangle\neq\delta(\theta-\phi)$.

$$\boxed{\int d(\bar\phi,\phi)\,e^{-\sum_i\bar\phi_i\phi_i}\,|\phi\rangle\langle\phi| = 1_F}$$ {identity in Fock space (17)

measure: $d(\bar\phi,\phi) \equiv \prod_i\left(\frac{1}{\pi}d\bar\phi_i\,d\phi_i\right) = \prod_i\frac{1}{\pi}d(\mathrm{Re}\,\phi_i)\,d(\mathrm{Im}\,\phi_i)$ (18)

(19)

Proof: If $[a_i,(17)] = [a_i^\dagger,(17)] = 0\ \forall i$, $\underset{\text{Schur's lemma}}{\text{then}\Rightarrow}$ $(17) = 1_F$

But

$$a_i\,(17) = \int d(\bar\phi,\phi)\,\underbrace{\underbrace{e^{-\sum_i\bar\phi_i\phi_i}\,\phi_i}_{-\partial_{\bar\phi_i}e^{-(\ )}}|\phi\rangle\langle\phi|}_{e^{-(\ )}\,\partial_{\bar\phi_i}|\phi\rangle\langle\phi|}$$ (20)

integrate by parts, boundary term vanishes!

$$= e^{-(\ )}|\phi\rangle\langle\phi|a_i = (17)\,a_i\ ✓$$ (21)

Similar for $a_i^\dagger$!

Normalization: $\langle 0|(17)|0\rangle = \prod_j\int\frac{d\bar\phi_j\,d\phi_j}{\pi}\,e^{-\bar\phi_j\phi_j} = \prod_i\left(\frac{\pi}{\pi}\right) = 1$ ✓ FF 6 (22)

Coherent-state representation (CSR) of general state $|\psi\rangle$: [Negele & Orland, p. 23]

$$|\psi\rangle = 1\,|\psi\rangle = (17)\,|\psi\rangle = \int d(\bar\phi,\phi)\,e^{-\sum_j\bar\phi_j\phi_j}\,|\phi\rangle\,\underbrace{\langle\phi|\psi\rangle}_{\equiv\,\psi(\bar\phi)}$$ (23)

$\psi(\bar\phi) = \langle\phi|\psi\rangle$ is the CSR of $|\psi\rangle$; $\psi(\bar\phi)$ is an analytic function of $\bar\phi$

Scalar product:

$$\langle\psi'|\psi\rangle \overset{(23)}{=} \int d(\bar\phi,\phi)\,e^{-\sum_j\bar\phi_j\phi_j}\,\underbrace{\langle\psi'|\phi\rangle\langle\phi|\psi\rangle}_{\psi'(\phi)\,\psi(\bar\phi)}$$ (24)

Action of a_i, $a_i^\dagger$ in CSR:

$$\langle\phi|a_i|\psi\rangle \overset{(13')}{=} \partial_{\bar\phi_i}\langle\phi|\psi\rangle = \partial_{\bar\phi_i}\psi(\bar\phi)$$ (25a)

$$\langle\phi|a_i^\dagger|\psi\rangle \overset{(10)}{=} \bar\phi_i\,\psi(\bar\phi)$$ (25b)

$\Rightarrow$ symbolically we may write: $\boxed{a_i = \partial_{\bar\phi_i}\ ,\qquad a_i^\dagger = \bar\phi_i}$ (26)

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Example: Schrödinger eq: $$\hat{H}(a_i^\dagger, a_i)|\psi\rangle = E|\psi\rangle \tag{27}$$

CSR: $\langle\phi|(27)$: $$H(\partial_{\bar\phi_i}, \bar\phi_i)\,\psi(\bar\phi) = E\,\psi(\bar\phi) \tag{28}$$

Matrix elements of normal-ordered operators:

$$\langle\phi|\,A(a_i^\dagger, a_i)\,|\phi\rangle \overset{(16)}{=} A(\partial_{\bar\phi_i}, \bar\phi_i)\, e^{\sum_j \bar\phi_j \phi_j} \tag{29}$$

CS are not number eigenstates; distribution of $\{n_i\}$:

$$|\langle n_1, n_2, \dots |\phi\rangle|^2 = |\langle n_1, n_2, \dots|\exp\left[\sum_j \phi_j a_j^\dagger\right]|0\rangle|^2 \tag{30}$$

$$= \left|\prod_i \frac{\phi_j^{n_j}}{\sqrt{n_j}}\right|^2 = \prod_i \frac{|\phi_j|^{2n_j}}{n_j!} = \text{Poisson-distribution} \tag{31}$$

FFI8

Average particle #: $$\bar N = \frac{\langle\phi|\sum_j a_j^\dagger a_j|\phi\rangle}{\langle\phi|\phi\rangle} \overset{(6)}{=} \sum_j \bar\phi_j \phi_j = \sum_j |\phi_j|^2 \tag{32}$$

Variance: $$\sigma^2 = \frac{\langle\phi|\,\hat N^2|\phi\rangle}{\langle\phi|\phi\rangle} - \bar N^2$$

$$= \langle\phi|\sum_{ij} a_i^\dagger \underbrace{a_i\, a_j^\dagger}_{a_j^\dagger a_i + \delta_{ij}} a_j|\phi\rangle/\langle\phi|\phi\rangle - \bar N^2$$

$$= \sum_{ij} \bar\phi_i \bar\phi_j \phi_i \phi_j + \sum_i \bar\phi_i \phi_i - \bar N^2$$

cancel

$$= \bar N$$

$\Rightarrow$ relative width: $$\frac{\sigma}{\bar N} = \frac{1}{\sqrt{\bar N}}$$

in thermodynamic limit $\bar N \to \infty$, CS becomes sharply peaked!