Gaussian Integrals

[see Altland, Simons, 3.2; Negele, Orland, Ch. 1]

[we discuss symmetric, Hermitian matrices; for general case, see standard references]

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Mother of all Gaussian integrals: $(x \in \mathbb{R})$

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + bx} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}} \quad (14)$$

$y = x - b/a$

$$\parallel \int_{-\infty}^{\infty} dy\, e^{-\frac{a}{2}y^2 + \frac{b^2}{2a}} \,/\!/$$

Multi-dim. version: (A: real, symm, pos.def.)

$$\boxed{\int dx_1 \ldots dx_n\, e^{-\frac{1}{2} x_i A_{ij} x_j + x_i J_i} = \frac{(2\pi)^{n/2}}{(\det A)}\, e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}} \quad (15)$$

let $y_i = x_i - A^{-1}_{ij} J_j$

$$= \underbrace{\int dy_1 \ldots dy_n\, e^{-\frac{1}{2} y_i A_{ij} y_j}}_{\parallel} \; e^{+\frac{1}{2} J_i A^{-1}_{ij} J_j} \quad (16)$$

$z_k = O^{-1}_{ki}\, y_i$ ← diagonalizes A

$$\int dz_1 \ldots dz_n\, e^{-\frac{1}{2}\sum_i a_i z_i^2} = \prod_n \frac{(2\pi)^{n/2}}{a_n}$$

a_i ← eigenvalues of A

$\Longrightarrow (15)$ ✓

Complex version, H has positive Hermitian part:

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$$\boxed{\int \prod_{i=1}^{n} \underbrace{\tfrac{1}{\pi}\, d(\mathrm{Re}\, x_i)\, d(\mathrm{Im}\, x_i)}_{d(\bar{x}_i, x_i)}\; e^{-\bar{x}_i H_{ij} x_j + \bar{J}_i x_i + J_i \bar{x}_i} = \frac{e^{\bar{J}_i H^{-1}_{ij} J_j}}{\det H}} \quad (17)$$

$$\text{since } \int d(\bar{x}_i, x_i)\, e^{-a\bar{x}_i x_i} = \frac{1}{a} \quad (18)$$

note the difference!

For Grassmann variables, analogous result is:

$$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta} a \eta} = \int d\bar{\eta}\, d\eta\, (1 - \bar{\eta} a \eta) = a \quad (19)$$

$$\boxed{\int \prod_i d\bar{\eta}_i\, d\eta_i\; e^{-\bar{\eta}_i H_{ij} \eta_j + \bar{\xi}_i \eta_i + \xi_i \bar{\eta}_i} = (\det H)\, e^{\bar{\xi}_i H^{-1}_{ij} \xi_j}} \quad (20)$$

where $\{\eta_i, \bar{\eta}_i, \xi_i, \bar{\xi}_i\}$ are GV. note: $\det H$ is upstairs!

For proof of (20), see Negele, Orland, p.35, Altland, Simons, p.165.

In brief: again make transf. to diagonalize H_{ij}. Use the identity:

if $\bar\xi_i = M_{ij}\bar\eta_j$, $\quad \xi_i = M'_{ij}\eta_j$, are linear transformations, (21)

then $$\int d(\bar\eta,\eta)\, f(\bar\eta,\eta) = \underbrace{\det(MM')}_{\text{inverse of usual Jacobian!!}} \int d(\bar\xi,\xi)\, f(\eta(\xi),\bar\eta(\bar\xi)) \qquad (22)$$

since $$\bar\xi_1\dots\bar\xi_n = \det M\, \bar\eta_1\dots\bar\eta_n, \qquad \xi_1\dots\xi_n = \det M'\, \eta_1\dots\eta_n \qquad (23)$$

Prefactor must be polynomial of order n in matrix elements M_{ij}, antisymmetric under interchange of $\bar\xi_i\,\bar\xi_{i+1}$ or $\eta_i\,\eta_{i+1}$ $\Rightarrow \det M$.

Field integral for quantum partition function

Partition function: $Z = \mathrm{tr}\, e^{-\beta(\hat H - \mu\hat N)}$ $\qquad \hat K = \hat H - \mu\hat N$ (24)

$$= \int d(\bar\psi,\psi)\, e^{-\sum_i \bar\psi_i\psi_i} \langle \zeta\psi| e^{-\beta\hat K}|\psi\rangle \qquad (25)$$

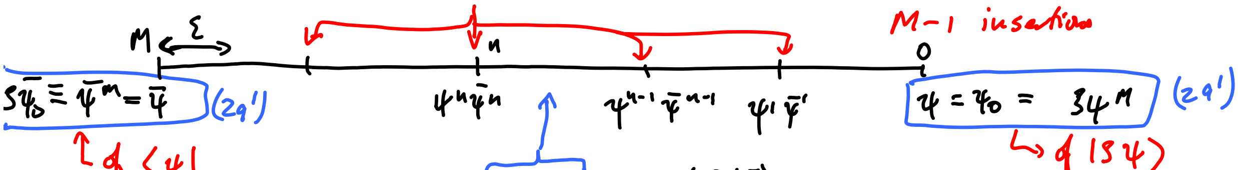
Time-slicing: $\beta = \varepsilon M$ $\qquad e^{-\beta\hat K} = \left(e^{-\varepsilon\hat K}\right)^M$

operator label i

time-slice label

$$1 = \int d(\bar\psi^n,\psi^n)\, e^{-\sum_i \bar\psi^n_i\psi^n_i} |\psi^n\rangle\langle\psi^n| \qquad (27)$$

M−1 insertions (28)



$$\langle\bar\psi^n| e^{-\varepsilon\hat K}|\psi^{n-1}\rangle \overset{(23.65)}{=} e^{-\varepsilon K(\bar\psi^n,\psi^{n-1})}\, e^{\sum_i \bar\psi^n_i\psi^{n-1}_i} \qquad (29)$$

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$$Z = \lim_{M\to\infty} \int \underset{d(\bar\psi^M, \psi^M)}{\underset{\shortparallel}{d(\bar\psi, \psi)}} \prod_{n=1}^{M-1} d(\bar\psi_n, \psi_n)\, e^{-S(\bar\psi, \psi)} \tag{30}$$

from completeness identity

from $\langle \psi^n | \psi^{n-1} \rangle$

$$S(\bar\psi, \psi) = \varepsilon \sum_{n=1}^{M} \sum_i \bar\psi_i^n \left\{ \frac{\psi_i^n - \psi_i^{n-1}}{\varepsilon} \right\} + K(\bar\psi_i^n, \psi_i^{n-1}) \tag{31}$$

$$\boxed{S(\bar\psi, \psi) =: \int_0^\beta d\tau \sum_i \left[\bar\psi_i(\tau)\, \partial_\tau\, \psi_i(\tau) + K(\bar\psi_i(t), \psi_i(t)\right]} \tag{32}$$

where we used symbolic trajectory notation: (as a shorthand for discrete expressions)

$$\{\psi_i^1, \ldots \psi_i^M\} =: \psi_i(t) \tag{33}$$

$$\boxed{Z = \int_{\substack{\psi_i(\beta) = \zeta \psi_i(0) \\ \bar\psi_i(\beta) = \zeta \bar\psi_i(0)}} \mathcal{D}(\bar\psi(\tau), \psi(\tau))\, e^{-S(\bar\psi, \psi)}} \tag{34}$$

this is shorthand for (29')

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Note:
integral is over complex numbers with periodic boundary cond. for bosons
Grassmann variables with anti-periodic boundary cond. for fermions

$$\text{E.g.} \quad \hat K(a^\dagger, a) = \sum_{ij} (h_{ij} - \mu_j \delta_{ij})\, a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} \underbrace{a_i^\dagger a_j^\dagger a_k a_l}_{\text{normal-ordered version}} \tag{35}$$

$$S(\bar\psi, \psi) = \int_0^\beta d\tau \left[\sum_{ij} \bar\psi_i(\tau) \left[(\partial_\tau - \mu)\delta_{ij} + h_{ij} \right] \psi_j(\tau) + \sum_{ijkl} V_{ijkl}\, \bar\psi_i(\tau)\, \bar\psi_j(\tau)\, \psi_k(\tau)\, \psi_l(\tau) \right] \tag{36}$$

Comments: $\bar\psi\, \partial_\tau \psi - H(\bar\psi, \psi)$ is analogous to $p\dot q - H(p, q)$ (37)

where $\psi, \bar\psi$ are eigenvalues of canonically conjugate operators $a_i, a_i^\dagger$

just as q, p " " " " " $\hat q, \hat p$

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(34),(36) are "time-representation" of Z. Now switch to "frequency representation"!

Boundary conditions imply: $\bar\psi_i(0) = \zeta\, \bar\psi_i(\beta)$, $\psi_i(0) = \zeta\, \psi_i(\beta)$ (38)

$\Rightarrow$ periodic/antiperiodic on $[0,\beta]$, so expand in Fourier series:

$$\begin{Bmatrix}\psi_i(\tau)\\ \bar\psi_i(\tau)\end{Bmatrix} = \frac{1}{\sqrt\beta}\sum_m \begin{Bmatrix}\psi_{im}\\ \bar\psi_{im}\end{Bmatrix} e^{\mp i\omega_m\tau}, \quad (39a) \qquad \begin{Bmatrix}\psi_{im}\\ \bar\psi_{im}\end{Bmatrix} = \frac{1}{\sqrt\beta}\int_0^\beta d\tau \begin{Bmatrix}\psi(\tau)\\ \bar\psi(\tau)\end{Bmatrix} e^{\pm i\omega_m\tau} \quad (39b)$$

$$\text{where} \quad \omega_m = \left\{\begin{matrix} 2m\pi/\beta & \text{bosons}, \\ (2m+1)\pi/\beta & \text{fermions}, \end{matrix} \quad m\in\mathbb{Z} \right\} \quad \text{"Matsubara frequency"} \quad (40)$$

Check: $\psi(\beta) = \sum_m \psi_{im}\, e^{i\pi\binom{2m}{2m+1}} = \pm\, \psi(0)$ ✓ (41)

$(39a)^{-1} = (39b)$, because: $\int_0^\beta d\tau\, e^{-i\omega_n\tau} = \beta\,\delta_{n,0}$ (42)

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Note: $\psi(\tau)$ is dimensionless, ψ_{im} has units of $\sqrt{\tau} = \frac{1}{\sqrt{\text{Energy}}}$ (42)

The Matsubara modes ψ_{im} carry same information as $\psi_i(\tau)$.

Hence, Z can be written in "frequency formulation":

$$Z = \int \mathcal{D}(\bar\psi,\psi)\, e^{-S(\bar\psi,\psi)}, \quad \text{where} \quad (43)$$

$$\int \mathcal{D}(\bar\psi,\psi) = \int \prod_m d(\bar\psi_m, \psi_m) = \int \prod_m \prod_i d(\bar\psi_{im}\, \psi_{im}) \quad (44)$$

for each Matsubara index m we have integration over a coherent state basis $\{|\psi_m\rangle\}$, with normalization: (45)

$$\int d(\bar\psi_{in}\, \psi_{in})\, e^{-\bar\psi_{in}\, \varepsilon\, \psi_{in}} = (\varepsilon\,)^{-\zeta}\beta \qquad \text{since } [\psi] = \text{energy}^{-1/2}$$

← to get dimensions right

Action becomes, in frequency representation:

$$S[\bar\psi, \psi] = \sum_{ij,m} \bar\psi_{im} (-i\omega_m - \mu)\delta_{ij} \psi_{jm} \quad + \tag{45a}$$

$$\frac{1}{\beta} \sum_{ijkl,\{\omega_{n_i}\}} V_{ijkl} \bar\psi_{im_1} \bar\psi_{jm_2} \psi_{km_3} \psi_{lm_4} \delta_{m_1+m_2, m_3+m_4} \tag{45b}$$

To sort out possible convergence problems, recall that $\bar\psi(\tau)$ is evaluated infinitesimally <u>later</u> than $\psi(\tau)$, so

$\bar\psi(\tau)\psi(\tau)$ stands for $\bar\psi(\tau + 0^+)\psi(\tau)$, so,

(45a) gets extra factor $e^{i0^+\omega_m}$

(45b) " " " $e^{i0^+(\omega_{m_1}+\omega_{m_2})}$