Correlators: (see Negele, Orland, 2.3) 20.11.08 FFI 42

Recall from lecture on Gaussian integrals (complex or Grassmann fields)

$$\mathcal{G}[\xi_i,\bar\xi_i] = \int d(\bar\psi,\psi)\, e^{-\sum_{ij}\bar\psi_i H_{ij}\psi_j + \sum_i(\bar\psi_i\xi_i + \bar\xi_i\psi_i)} \qquad (1)$$

$$= [\det H]^{-\zeta}\, e^{\sum_{ij}\bar\xi_i H^{-1}_{ij}\xi_j} = \text{"generator of correlation functions"} \qquad (2)$$

$$\langle\psi_i\bar\psi_j\rangle \equiv \frac{\int d(\bar\psi,\psi)\,\psi_i\bar\psi_j\, e^{-\sum_{ij}\bar\psi_i H_{ij}\psi_j}}{\int d(\bar\psi,\psi)\, e^{-\sum_{ij}\bar\psi_i H_{ij}\psi_j}} \qquad (3)$$

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$$\langle\psi_i\bar\psi_j\rangle \overset{(1)}{=} -\frac{1}{\mathcal{J}[0,0]}\,\frac{\delta^2\mathcal{J}[\xi_i,\bar\xi_i]}{\delta\bar\xi_i\,\delta\xi_j}\bigg|_{\xi_i=\bar\xi_j=0} = H^{-1}_{ij} \qquad (4)$$

Similarly:

$$\langle\psi_{i_1}\cdots\psi_{i_n}\bar\psi_{j_n}\cdots\bar\psi_{j_1}\rangle = \frac{(\zeta)^n}{\mathcal{G}[0,0]}\,\frac{\delta^{2n}\mathcal{G}[\xi_i,\bar\xi_i]}{\delta\bar\xi_{i_1}\cdots\delta\bar\xi_{i_n}\,\delta\xi_{j_n}\cdots\delta\xi_{j_1}}\bigg|_{\xi_i=\bar\xi_i=0} \qquad (5)$$

$$= \sum_P \zeta^P H^{-1}_{i_1 j_{P_1}}\cdots H^{-1}_{i_n j_{P_n}}$$

Wick's theorem (requires quadratic action)

Explicitly, for $n=2$:

$$\langle\psi_1\psi_2\bar\psi_2\bar\psi_1\rangle = (-)^2\,\frac{\delta^4\left[e^{\bar\xi_i H^{-1}_{ij}\xi_j}\right]}{\delta\bar\xi_1\,\delta\bar\xi_2\,\delta\xi_2\,\delta\xi_1} \qquad (6)$$

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$$= (-)^2 \frac{1}{\delta\bar\xi_1\,\delta\bar\xi_2}\,\delta^2 \Big[(-)\,\bar\xi_j H^{-1}_{j2} \;\; (-)\,\bar\xi_i H^{-1}_{i1} \;\; e^{(\;\;)} \Big|_{\xi_i = \bar\xi_j = 0} \qquad (7)$$

$$= H^{-1}_{22}\, H^{-1}_{11} + \zeta\, H^{-1}_{21}\, H^{-1}_{12} = \langle \psi_1\, \psi_2\, \bar\psi_2\, \bar\psi_1 \rangle \qquad (8)$$

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Example (A.S., 4.5.4)

Consider free 1-D fermions; with $H = \int dx \;\bar\psi\,(\mp i\partial_x)\,\psi$ (9)

Action: $S = \int dx \int d\tau \;\bar\psi\,(\partial_\tau \mp i\partial_x)\,\psi$ (10)

Consider correlator:

$$G_\pm(x,\tau) = Z_\pm^{-1} \int \mathcal{D}(\bar\psi,\psi)\; \psi(x,\tau)\,\bar\psi(0,0)\; e^{-S_\pm[\bar\psi,\psi]} \qquad (11)$$

$$= (\partial_{\tau'} \mp i\partial_{x'})^{-1}\Big|_{(x,\tau;0,0)}$$

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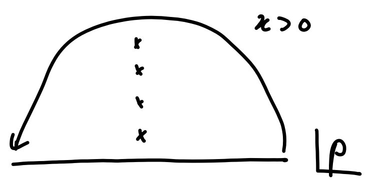
make this explicit in terms of eigenvalues of the operator

$$= \frac{1}{\beta}\sum_n \underbrace{\frac{1}{L}\sum_p}_{\int \frac{dp}{2\pi}} \frac{e^{+ipx - i\omega_n\tau}}{-i\omega_n \pm p} \qquad \text{(assume } x>0\text{)} \qquad (12)$$



$$= \mp \underbrace{\frac{i}{\beta}\sum_n}_{\int d\omega} \Theta(\pm n)\; e^{\omega_n(-i\tau \mp x)} \qquad (13)$$

$$T \to 0 = \mp i \int_0^\infty \frac{d\omega}{2\pi}\; e^{\omega(\mp i\tau - x)} = \frac{1}{2\pi}\,\frac{1}{\tau \mp ix} \qquad (14)$$

Exercise: 4.5.4 b: Boson correlators

<u>Free propagators:</u> $H - \mu N = \sum_a \xi_a c_a^\dagger c_a$

$$-\mathcal{G}_a^>(\tau) \equiv \langle \psi_a(\tau)\, \psi_a^\dagger(0) \rangle = e^{-\tau\xi_a} \underbrace{\langle c_a c_a^\dagger \rangle}_{1 + \zeta \langle c_a^\dagger c_a \rangle} \tag{15}$$

$$= e^{-\tau\xi_a} \left(1 + \frac{\zeta}{e^{\beta\xi_a} - \zeta} \right)$$

$$= e^{-\tau\xi_a} \frac{1}{1 - \zeta e^{-\beta\xi_a}} \tag{16}$$

$$\mathcal{G}_a^>(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\, \mathcal{G}_a^>(\tau) \overset{(16)}{=} - \int_0^\beta d\tau\, \frac{e^{\tau(i\omega_n - \xi_a)}}{1 - \zeta e^{-\beta\xi_a}}$$

$$= \frac{1}{i\omega_n - \xi_a} (-) \frac{e^{\beta i\omega_n - \xi_a} - 1}{1 - \zeta e^{-\beta\xi_a}} = \frac{1}{i\omega_n - \xi_k} \tag{17}$$

<u>Redo with functional integral:</u> $S = \int d\tau \sum_a \bar{\psi}_a (\partial_\tau + \xi_a) \psi_a$

$$-\mathcal{G}_a^>(\tau) = \int d(\bar{\psi}, \psi)\, \langle \psi_a(\tau)\, \bar{\psi}(0) \rangle = (\partial_{\tau'} + \xi_a)^{-1} \Big|_{(\tau, 0)} \tag{18}$$

$$= \frac{1}{\beta} \sum_n \frac{e^{-i\omega_n\tau}}{-i\omega_n + \xi_a} \tag{19}$$

$$\mathcal{G}_a^>(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau} = \frac{1}{i\omega_n - \xi_a} \quad \checkmark . \tag{20}$$

Problem: AS. 4.5.5 : Frequency summations

AS. 4.5.6 : Pauli paramagnetism

Electron-Phonon interaction (A.S., 4.5.7)

$$H_{el\text{-}ph} = \gamma \int d^d r \; \hat{n}(\vec{r}) \; \vec{\nabla} \cdot \vec{u}(\vec{r}) \qquad \text{(neglect spin for simplicity)} \qquad (21)$$

Electrons sense induced charge $\rho_{ind} \sim \vec{\nabla} \cdot \vec{P}$, where $\vec{P} \sim \vec{u}$ is polarization generated by lattice distortion

$$\int d\vec{r} \, e^{-i\vec{q}\cdot\vec{r}} \, \vec{u}(\vec{r}) = \vec{u}_{\vec{q}} = \sum_j \vec{e}_j \left(a_{qj} + a^\dagger_{-qj}\right) \frac{1}{\sqrt{2m\omega_q}} \qquad (22)$$

↑ unit vector in j-direction.

$$\Rightarrow \quad H_{el\text{-}ph} = \gamma \sum_{\vec{k},\vec{q},j} \frac{i q_j}{\sqrt{2m\omega_q}} \hat{n}_q \left(a_{\vec{q}j} + a^\dagger_{-\vec{q},j}\right) \qquad (23)$$

where $\hat{n}_q = \sum_{\vec{k}} c^\dagger_{\vec{k}+\vec{q}} c_{\vec{k}}$ (24)

Coherent-State action?

$$Z = \int \mathcal{D}(\bar{\psi},\psi) \, \mathcal{D}(\bar{\phi},\phi) \, e^{-S_{el}(\bar{\psi},\psi) - S_{ph}(\bar{\phi},\phi) - S_{el\text{-}ph}[\bar{\psi},\psi,\bar{\phi},\phi]} \qquad (25)$$

electrons Grassmann   phonons complex

Action in frequency-momentum representation:

$$S_{ph} = \sum_{qj} \bar{\phi}_{qj} \left(-i\omega_n + \omega_q\right) \phi_{qj} \qquad q \equiv (\omega_n, \vec{q}) \qquad (26)$$

$$S_{el\text{-}ph} \overset{(23)}{=} \gamma \sum_{qj} \frac{i q_j}{(2m\omega_q)^{1/2}} \rho_q \left(\phi_{qj} + \bar{\phi}_{-qj}\right) \qquad (27)$$

q_j: j-component of $\vec{q}$

where $\rho_q = \sum_k \bar{\psi}_{k+q} \psi_k \qquad k = (\omega_m, \vec{k}) \qquad (28)$

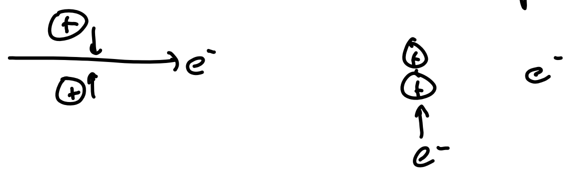
S_{el} : need not be specified explicitly!

"Integrate out phonons" $\Longleftrightarrow$ do $\int \mathcal{D}(\bar{\phi},\phi)$ integral :

$$\int \mathcal{D}(\bar{\phi},\phi)\; e^{-S_{ph} - S_{el\text{-}ph}} = e^{\underbrace{\sum_q \left(\frac{\vec{q}^{\,2}}{2m\omega_q}\right)\frac{1}{-i\omega_n + \omega_q}\,\rho_q\,\rho_{-q}}_{-S_{el\text{-}el}}}$$

$\Rightarrow$ phonons induce electron-electron interaction!

(electron attracts ions, this creates phonon, and other electrons follow)



: retarded interaction!

$$S_{el\text{-}el} = -\frac{\gamma^2}{2m}\sum_{\substack{q\\ n>0}} \frac{\vec{q}^{\,2}}{\omega_n^2 + \omega_q^2}\,\rho_q\,\rho_{-q}$$

Sloppily transform back from Matsubara to real frequencies, $\omega_n \to -i\omega$

$$S_{el\text{-}el} \sim -\frac{1}{-\omega^2 + \omega_q^2} < 0 \quad \text{if} \quad \omega < \omega_q$$

$\Rightarrow$ we have <u>attractive</u> interaction at low frequencies!!

(origin of superconductivity).