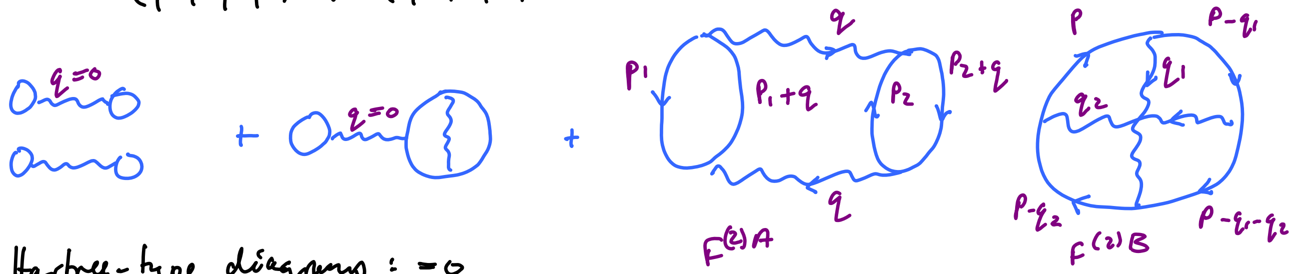
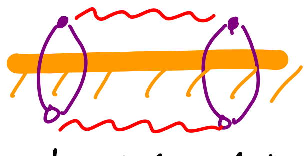
## 2nd order processes 

$$(\bar\psi\psi\bar\psi\psi)\, V\, (\bar\psi\psi\bar\psi\psi)\, V$$



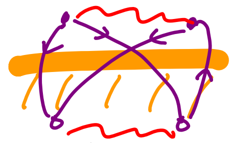

Hartree-type diagrams: $= 0$

$$F^{(2)A} = -\frac{T^3}{L^6} \sum_{p_1 p_2 q} \mathcal{G}_{p_1} \mathcal{G}_{p_1+q} \mathcal{G}_{p_2} \mathcal{G}_{p_2+q}\, V^2(q) \qquad (1)$$



two independent p-h excitations interact

$$F^{(2)B} = \frac{T^3}{2L^6} \sum_{p q_1 q_2} \mathcal{G}_p \mathcal{G}_{p-q_1} \mathcal{G}_{p-q_1-q_2} \mathcal{G}_{p-q_2} V(q_1) V(q_2) \qquad (2)$$



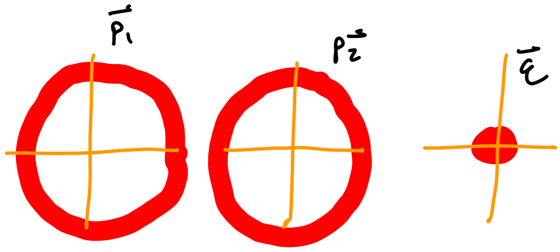
one p-h excitation <u>self</u>-interacts

In general: large contributions if arguments of all GF satisfy $|p| \sim p_F$ 

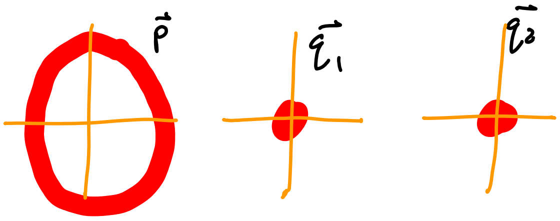
[ (intuitively: transport is carried by exc. close to $\varepsilon_F$)
technically: $\mathcal{G}^{(0)}_{\tilde p} \sim \dfrac{1}{i\omega_n - (\varepsilon_{\vec p} - \mu)}$ ]

When does this happen?

$F^{(2)A}$: if $\vec q \simeq 0$, near Fermi surfaces of both $p_1$ and $p_2$



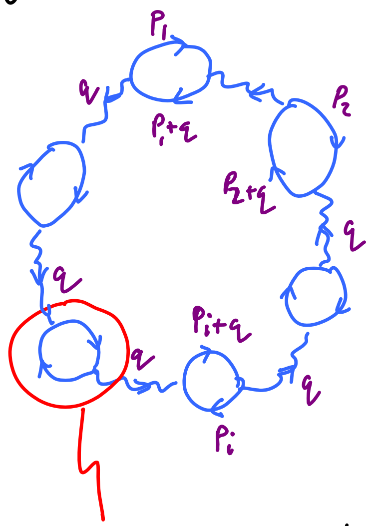

$F^{(2)B}$: if $\vec q_1 \approx 0$ and $\vec q_2 \simeq 0$, near Fermi surface of $p$




Hence $\dfrac{F^{(2)B}}{F^{(2)A}} \gg 1$ by a factor $(\text{surface of Fermi sphere})^{-1} \sim (\text{density})^{-1} \sim r_s$

PT10

By analogy, dominant process in any order of pert. theory has "ring graph" structure, with maximal number of independent particle-hole excitations:




$$F^{(n)}_{RPA} = \frac{T}{2} \sum_q \frac{1}{n}\Big(-V(q) \underbrace{\frac{2T}{L^3} \sum_p G_p\, G_{p+q}}_{\Pi_q}\Big)^n \qquad (1)$$

$\Pi_q$ = "polarization operator"

"polarization bubble" since p-h-exc. polarizes medium

$\frac{1}{n} = \frac{(n-1)!}{n!}$ ← ways to arrange $S_{int}$ operators in a circle. ($n!$ ← overall prefactor)

RPA : "Random phase approximation"

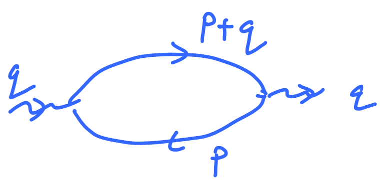
since "random phase of p-h-exc." gets neutralized after each polarization bubble, (other processes carry a random phase that averages to zero).

PT11

Free energy in RPA :

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n \qquad (1)$$

$$F_{RPA} = \sum_n F^{(n)}_{RPA} = \frac{T}{2} \sum_q \ln\big(1 - V(\vec q)\, \Pi_q\big) \qquad (2)$$




$$\Pi_{\vec q, \omega_n} \overset{(10.1)}{=} \frac{2T}{L^3} \sum_p G_p\, G_{p+q}$$

$$= \frac{2T}{L^3} \sum_{\vec p, \omega_m} \frac{1}{i\omega_n - \xi_p} \; \frac{1}{i\omega_{n+m} - \xi_{p+q}} \qquad (3)$$

$\xi_p = \varepsilon_p - \mu = \frac{p^2}{2m} - \mu$

Exercise: do Matsubara sum!

$$= \frac{2}{L^3} \sum_{\vec p} \frac{\overbrace{n_F(\varepsilon_{p+q}) - n_F(\varepsilon_p)}^{\frac{1}{m}\vec p\cdot\vec q\, \partial_{\varepsilon_p} n(\varepsilon_p) \xrightarrow{T\to 0} -\delta(\varepsilon_p-\mu)}}{i\omega_m + \underbrace{\xi_{p+q} - \xi_p}_{\frac{1}{m}\vec p\cdot\vec q \,+\, O(\frac{q^2}{m})}} \qquad (4)$$

$O(\frac{q^2}{m})$ ← negligible compared to $q/p_F$

$V(q) = \frac{e^2}{q^2}$ is largest for $q \to 0$, hence expand in powers of $|q|/p_F$ :

Doing the $\bar{p}$-integral, yields, for isotropic, 3D medium, the "Lindhardt function": $\boxed{\text{PT12}}$

$$\Pi_{\bar{q},\omega_n} = -\nu_0 \left[ 1 - \frac{i\omega_m}{v_F q} \ln\left( \frac{i\omega_m + v_F q}{i\omega_m - v_F q} \right) \right] \tag{1}$$

where $$\nu_0 = \frac{1}{L^d} \sum_{\bar{p},\sigma} \delta(\mu - \varepsilon_p) = \frac{m p_F}{\pi^2} = \text{free DOS} \tag{2}$$

Homework: check (12.1)!!

Insert (12.1) into (11.2) and perform sum (this is complicated!)

$$\frac{F_{RPA}^{(n\geq 2 \text{ contributions})}}{N E_{Ryd}} = -0.142 + 0.0622 \ln r_s \tag{3}$$

(Gell-Mann & Brückner, 1957)

Collecting results:

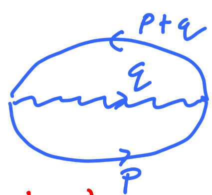
$$F = F_0 + F^{(1)} + F_{RPA}^{(n\geq 2 \text{ contributions})} + \dots \tag{4}$$

$$\sim N\left( \frac{1}{r_s^2} + \frac{1}{r_s} + 1 + \ln r_s + \dots \right) \tag{5}$$

RPA encodes screening. To see this, compare contributions $\boxed{\text{PT13}}$

of $F^{(1)}$ and $F_{RPA}$ to density: $$N = -\partial_\mu F \tag{1}$$

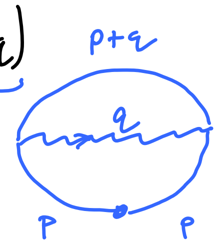
$$F^{(1)} = -\frac{T}{2} \sum_q V(q) \underbrace{\frac{2T}{L^3} \sum_p \mathcal{G}_p \mathcal{G}_{p+q}}_{\Pi_q} \tag{2}$$




$\partial_\mu \mathcal{G}_p = -\mathcal{G}_p^2$

(set $q \to -q$ in 2nd term)

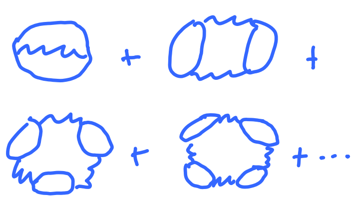
$$N^{(1)} = -\partial_\mu F^{(1)} = \frac{T}{2} \sum_q V(q) \underbrace{\left( -\frac{2T}{L^3} \sum_p 2\, \mathcal{G}_p^2 \mathcal{G}_{p+q} \right)}_{\partial_\mu \Pi_q} \tag{3}$$




$\rho^{(1)} = \partial_\mu N^{(1)}$ would lead to incorrect result $(\rho^{(0)} + \rho^{(1)})(\varepsilon_F)$ found earlier. (4)

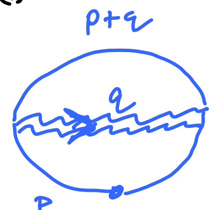
Consider instead:

$$F_{RPA}^{(\text{co.1})} = \frac{T}{2} \sum_q \ln(1 - V(q)\Pi_q) \tag{5}$$



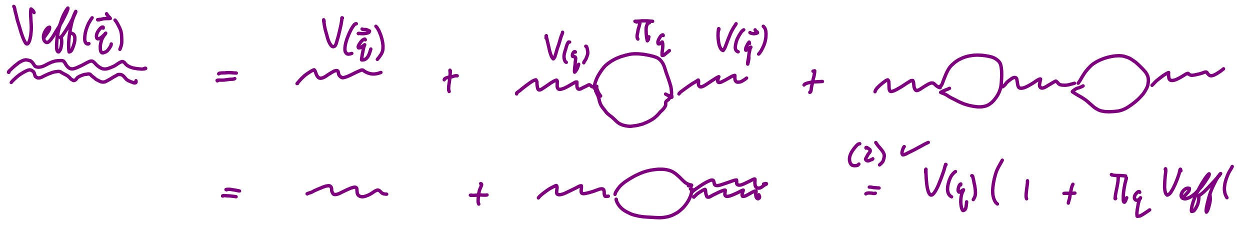
$$N_{RPA} = -\partial_\mu F_{RPA} = \frac{1}{2}\sum_q \boxed{\frac{V(\vec q)}{1 - V(q)\,\Pi_q}}\;\partial_\mu \Pi_q \qquad (1)$$

$\equiv V_{eff}(\vec q)$




where "effective int":

$$V_{eff}(\vec q) \equiv \frac{V(\vec q)}{1 - V(\vec q)\,\Pi_q} = V(\vec q)\left(1 + \Pi_q V_q + \Pi_q V_q \Pi_q V_q + \dots\right) \qquad (2)$$




$$\overset{(2)}{=} V(q)\left(1 + \Pi_q V_{eff}(\vec q)\right) \qquad (3)$$




Relation to screening:

Recall : electric field in vacuum medium

$$\vec D(\vec q,\omega) = \varepsilon(\vec q,\omega)\,\vec E(\vec q,\omega) \qquad (1)$$

since $\vec E = -\vec\nabla V_{eff}$ :
$\vec D = -\vec\nabla V$

$$\cancel{(-i\vec q)}\,V(\vec q) = \varepsilon(\vec q,\omega)\,\cancel{(-i\vec q)}\,V_{eff}(\vec q) \qquad (2)$$

where $\varepsilon(\vec q,\omega) = 1 + 4\pi\,\chi(\vec q,\omega)$ dielectric function (3)

$\chi(\vec q,\omega)$ = electromagnetic susceptibility

(19) $\Rightarrow$

$$V_{eff}(\vec q) = \frac{V(\vec q)}{\varepsilon(q)}, \qquad (4)$$

comparing (15.4) = (14.2), we identify:

$$\varepsilon(\vec q,\omega) \equiv 1 - V(\vec q)\,\Pi_q \qquad (5)$$

$$\chi(\vec q,\omega) = -\frac{1}{4\pi}V(\vec q)\,\Pi_q \qquad (6)$$

For explicit results, we need Lindhardt function $\Pi_{\vec{q},\omega_n} = (12.1)$ $\boxed{\text{PT16}}$

Important limits: compare two length scales:

$q^{-1}$ : "wavelength" of external perturbation (1)

$v_F/\omega$ : distance travelled by excitation at speed $v_F$ in time $\omega^{-1}$ (2)

consider "small frequency" limit: $\frac{v_F}{\omega} \gg q^{-1}$ : electron gas has time to adjust to ext. perturbation, to screen out fluctuations that tend to violate charge neutrality.

$\omega \ll v_F q$ : $\Pi(\vec{q}, 0) = -\nu_0$ (3)

In extreme ("static") limit: $V_{eff}(q) = \frac{1}{V^{-1}(\vec{q}) + \nu_0} = \frac{4\pi e^2}{q^2 + \lambda^{-2}}$

where $\lambda = (4\pi e^2 \nu_0)^{-1/2}$ = Thomas-Fermi screening length

cures IR problems!!

$\Rightarrow$ Fourier-transf: $V_{eff}(\vec{r}) = e^2 \frac{e^{-|\vec{r}|/\lambda}}{|\vec{r}|}$ = short-ranged! (4)

$\boxed{\text{PT17}}$

Thomas-Fermi-screening: describes density response to a "slowly varying" external perturbation $V(\vec{r})$. It induces effective interaction $V_{eff}(\vec{r})$, which shifts energy levels $\varepsilon_p \to \varepsilon_p - V_{eff}(\vec{r})$

(simultaneous use of $\vec{p}$ and $\vec{r}$ labels OK only if $V_{eff}$ is slowly varying)

Induced density:

$$\rho_{ind}(\vec{r}) = -e \int \frac{d^3p}{(2\pi)^3} \left[ n_F(\varepsilon_p - V_{eff}(\vec{r})) - n_F(\varepsilon_p) \right] \tag{1}$$

$$= e\, V_{eff}(\vec{r}) \int \frac{d^3p}{(2\pi)^3} \underbrace{\partial_\varepsilon n_F(\varepsilon)}_{\text{at low } T:\ -\delta(\varepsilon - \mu)} = -e V_{eff}\, \nu_0 \tag{2}$$

DOS at $\varepsilon_F$

Poisson: $\nabla^2 V_{eff}(\vec{r}) = -4\pi e\, \rho(\vec{r}) = -4\pi e \left( e\,\delta(\vec{r}) + \rho_{ind}(\vec{r}) \right)$ (3)

FT, use (28), to find $(q^2 + 4\pi e^2 \nu_0) V_{eff} = 4\pi e^2 = (4)$ ✓

Consider large-frequency limit: $v_F/\omega \ll q^{-1}$ ("dynamic polarization") $\boxed{\text{PT18}}$

Expand (12.1) to leading order in $v_F q/\omega \ll 1$:

$$V_{\text{eff}}(\vec{q}, \omega_n) = \frac{4\pi e^2}{q^2} \, \frac{1}{1 + \omega_p^2/\omega_n^2} \xrightarrow{i\omega_n \to \omega} \frac{4\pi e^2}{q^2} \, \frac{1}{1 - \omega_p^2/\omega^2} \qquad (1)$$

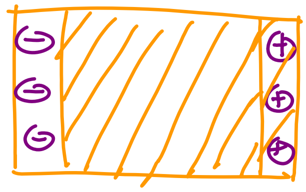
collective singular response at $\omega = \omega_p$

with $\omega_p = (4\pi n e^2/m)^{1/2}$ = "plasma frequency"

$n = N/L^3 = k_F^3/3\pi^2$ = density

Origin of plasmon mode:

displace electrons uniformly by $x$ against ions:



$\Rightarrow$ surface charge densities $\rho_\pm = \pm e x n$ (2)

cause electric field $E = -4\pi e n x$, producing force

$$\Rightarrow \quad m\ddot{x} = -eE = -\underbrace{4\pi e^2 n}_{\omega_p} \, x \quad \Rightarrow \boxed{\text{collective oscillations}} \qquad (3)$$

$=$ plasma frequency!