BCS1

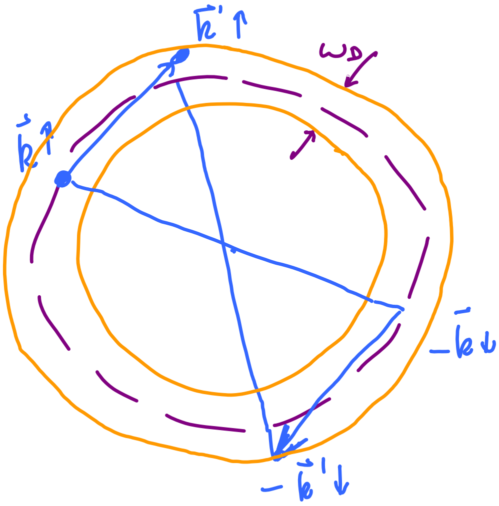
Superconductivity from field integral (Altland & Simons, 6.4.4)

Reading: 6.4.1 Basic concepts of BCS theory

$$\hat{H}_{BCS} = \sum_{\vec{k}\sigma} \varepsilon_{\vec{k}}\, n_{\vec{k}\sigma} - \frac{g}{L^d} \sum_{kk'q} c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} c_{-k'+q\downarrow} c_{k'\uparrow} \quad (1)$$

induced by phonons

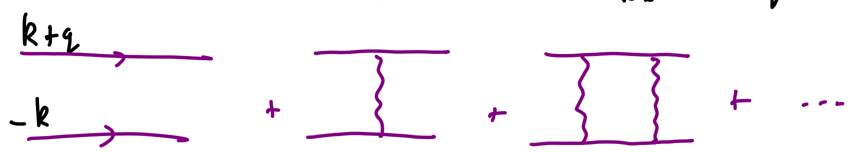
most phase space for $\vec{q}=0$



Reading: 6.4.2 Cooper instability:

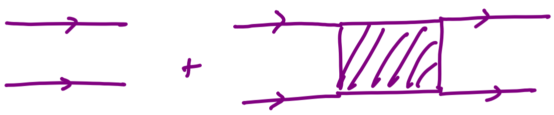
$$C(\vec{q},\tau) = \frac{1}{L^{2d}} \sum_{\vec{k}\vec{k}'} \langle \bar{\psi}^\dagger_{k+q\uparrow}(\tau)\, \bar{\psi}^\dagger_{-k\downarrow}(\tau)\, \psi_{k'+q\downarrow}(0)\, \psi_{-k'\uparrow}(0) \rangle \quad (2)$$

$C =$



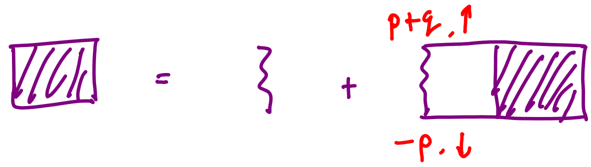
BCS2

$C =$



(1)

$\Gamma =$



(2)

$$\Gamma_q = g + g \frac{T}{L^d} \sum_p \mathcal{G}_{p+q} \mathcal{G}_{-p} \Gamma_q \quad (3a) \quad \Rightarrow \quad \Gamma_q = \frac{g}{1 - g \frac{T}{L^d} \sum_p \mathcal{G}_{p+q} \mathcal{G}_{-p}} \quad (3b)$$

$$\Gamma(\vec{q}=0, \omega_n = 0) \simeq \quad \vec{q}\to 0 \quad \nu \int_{-\omega_D} d\varepsilon\, \frac{d\varepsilon}{2\varepsilon}(1 - 2f(\varepsilon)) \approx -\nu \int_{-\omega_D}^{-T} d\varepsilon\, \frac{d\varepsilon}{\varepsilon} = \nu \ln \frac{\omega_D}{T} \quad (4)$$

$$\boxed{\Gamma_{(0,0)} = \frac{g}{1 - g\nu \ln \frac{\omega_D}{T}}} \quad (5)$$

- Perturbation theory (vertex) explodes when

$$T \to T_c = \omega_D\, e^{-\frac{1}{g\nu}} \quad (6)$$

$\Rightarrow$ Reason: calculation started from wrong reference state.

Reading: 6.4.3 Mean-field theory of superconductivity

BCS3

BCS: propose $\Delta = \frac{g}{L^d} \sum_{k} \langle \Omega_s | \overbrace{c_{-k\downarrow} c_{k\uparrow}}^{\text{"bosonic pair}} | \Omega_s \rangle \neq 0$ (1)

as mean-field order parameter

ground state (fermion coherent state)
(condensate of pairs)

$\hat{H}^{MF}_{BCS}$ = obtained by writing

$$\sum_{k} c^\dagger c^\dagger \sum_{k'} c c = \left[\left(\sum c^\dagger c^\dagger - \frac{\bar{\Delta} L^d}{g} \right) + \frac{\bar{\Delta} L^d}{g} \right] \left[\left(\sum c c - \frac{\Delta L^d}{g} \right) + \frac{\Delta L^d}{g} \right] \quad (2)$$

ignore terms quadratic in small fluctuations

$$\hat{H}^{MF}_{BCS} - \mu \hat{N} = \sum_{k} \left[\underbrace{\xi_k}_{(\varepsilon_k - \mu)} c^\dagger_{k\sigma} c_{k\sigma} - \left(\bar{\Delta}\, c_{-k\downarrow} c_{k\uparrow} + h.c. \right) + \frac{L^d |\Delta|^2}{g} \right] \quad (3)$$

= Bogoliubov-de Gennes or Gorkov Hamiltonian.

BCS4

$$= \sum_{k} \psi_k^\dagger \begin{pmatrix} \xi_k & -\Delta \\ -\bar{\Delta} & -\xi_k \end{pmatrix} \psi_k + \sum_{k} \xi_k + \frac{L^d |\Delta|^2}{g} \quad (1)$$

where Nambu spinor $\psi_k^\dagger = (c^\dagger_{k\uparrow}, c^\dagger_{-k\downarrow})$, $\psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{pmatrix}$ (2)

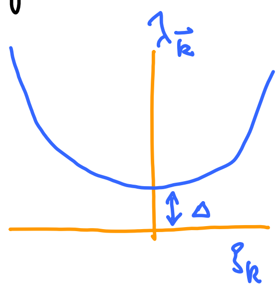
Diagonalize: $\chi_k = \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^\dagger_{-k\downarrow} \end{pmatrix} = U_k \psi_k$, (3)

then $\hat{H}^{MF}_{BCS} - \mu \hat{N} = \sum_{k\sigma} \lambda_k \alpha^\dagger_{k\sigma} \alpha_{k\sigma} + \sum_{k} (\xi_k - \lambda_k) + \frac{\Delta^2 L^2}{g}$ (4)

quasi-particle {operators / energies $\rightarrow \lambda_k = (\Delta^2 + \xi_k^2)^{1/2}$ (5)

(3) in (3.1) gives gap equation: (at T=0) $\Delta = \frac{g}{2L^d} \sum_{k} \frac{\Delta}{\sqrt{\Delta^2 + \xi_k^2}}$ (6)

Solution $\Delta \simeq 2\omega_D \exp\left(-\frac{1}{g\nu}\right)$ (7)



BCS5

Superconductivity from field integral

Action corresponding to real-space version of (1) :

[needed, since charged particles couple to EM field]

"minimal coupling"

$$S[\bar\psi,\psi] = \int_0^\beta d\tau \int d^d r \Big\{ \bar\psi_\sigma(\vec r)\Big[\partial_\tau + ie\phi + \frac{1}{2m}\big(-i\vec\nabla - e\vec A\big)^2 - \mu\Big]\psi_\sigma(\vec r)$$

Grassmann

$$- g\, c^\dagger_\uparrow(\vec r)\, c^\dagger_\downarrow(\vec r)\, c_\downarrow(\vec r)\, c_\uparrow(\vec r)\Big\} \qquad (1)$$

S has local gauge invariance under:

$$\psi \to e^{i\theta}\psi, \quad \bar\psi \to e^{-i\theta}\bar\psi, \quad \phi \to \phi - \frac{\partial_\tau\theta}{e}, \quad \vec A \to \vec A + \frac{\vec\nabla\theta}{e} \qquad (2)$$

Decouple by HS: $\quad \exp\Big\{ g \int d\tau\, d^d r\ \bar\psi_\uparrow \bar\psi_\downarrow \psi_\downarrow \psi_\uparrow \Big\} = \qquad (3)$

$$\int \mathcal{D}(\bar\Delta,\Delta)\, \exp\Big\{ -\int d\tau\, d^d r \Big[\frac{1}{g}|\Delta|^2 - \big(\bar\Delta \psi_\downarrow\psi_\uparrow + \Delta\bar\psi_\uparrow\bar\psi_\downarrow\big) \Big\}$$

complex field

BCS6

more compactly, in terms of Nambu spinor (compare 4.2)

$$\bar\Psi = (\bar\psi_\uparrow, \psi_\downarrow), \qquad \Psi = \begin{pmatrix}\psi_\uparrow \\ \bar\psi_\downarrow\end{pmatrix} \qquad (1)$$

$$S \overset{(5.3)}{=} \int \mathcal{D}(\bar\psi,\psi) \int \mathcal{D}(\bar\Delta,\Delta)\, \exp\Big\{ -\int d\tau \int d^d r \Big[\frac{1}{g}|\Delta|^2 - \underbrace{\bar\Psi\, \hat{\mathcal{G}}^{-1}\, \Psi}_{} \Big] \Big\} \qquad (2)$$

Gorkov GF: $$\hat{\mathcal{G}}^{-1} = \begin{bmatrix} \big(\hat{\mathcal{G}}_0^{(p)}\big)^{-1} & \Delta \\ \Delta & \big(\hat{\mathcal{G}}_0^{(h)}\big)^{-1} \end{bmatrix} \qquad (3)$$

[spinor · matrix · spinor multiplication in Nambu space]

with $$\Big[\hat{\mathcal{G}}_0^{(p/h)}\Big]^{-1} = -\partial_\tau \mp ie\phi \mp \frac{1}{2m}\big(\mp i\vec\nabla - e\vec A\big)^2 \pm \mu \qquad (4)$$

[extra minus for $\mathcal{G}_h$ from $\bar\psi\psi$ another minus from integrating by parts]

$$= -\partial_\tau + \begin{cases} -\hat H + \mu \\ +\hat H^T - \mu \end{cases} \qquad \big(\vec\nabla^T = -\vec\nabla\big) \qquad (5)$$

where $\hat H^T(\hat x, \hat p) = \hat H(\hat x, -\hat p)$ (1) $\boxed{\text{BCS7}}$

quantum transposition, or time-reversal operation.

$\Rightarrow$ hole energy $=$ $-$ particle energy, and

hole = particle travelling backwards in time. (2)

(6.2) is linear in $\bar\psi$, so integrate out ψ, in usual manner:

contains Δ! (see 6)

$$Z = \int \mathcal{D}(\bar\Delta, \Delta) \exp\left\{ -\frac{1}{g}\int d\tau\, d^d r\, |\Delta|^2 + \ln\det \hat{\mathcal{G}}^{-1} \right\} \qquad (3)$$

from reexponentiating the prefactor $\det \hat{\mathcal{G}}^{-1}$

from $\partial_{\bar\Delta} \mathcal{G}^{-1} = \partial_{\bar\Delta}\begin{pmatrix} \mathcal{G}_0^{-1} & \Delta \\ \bar\Delta & \bar{\mathcal{G}}_0^{-1} \end{pmatrix}$

Mean field: $\qquad g^{-1}\,\bar\Delta(\vec x, \tau) = \mathrm{tr}\left[\hat{\mathcal{G}}(\vec x, \tau; \vec x, \tau)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$ (4)

(compare p. BS9)

(homework: check this!)

trace over matrix indices in Nambu space

Assume homogeneous solutions: $\Delta(\vec x, \tau) = \Delta_0 = \text{const}$. (1) $\boxed{\text{BCS8}}$

In absence of any fields, $\vec A, \phi$, we get (in $\vec p, \omega_n$ representation)

$$\frac{\Delta_0}{g} = \frac{T}{L^d}\sum_{\vec p, n} \mathrm{Tr}\left\{\begin{pmatrix} i\omega_n - \xi_p & \Delta_0 \\ \Delta_0 & i\omega_n + \xi_p \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\} \qquad (2)$$

$$= \frac{T}{L^d}\sum_{\vec p, n}\frac{\bar\Delta_0}{\omega_n^2 + \underbrace{\xi_{\vec p}^2 + |\Delta_0|^2}_{= \lambda_{\vec p}^2 \text{ (see 4.5)}}} \qquad (3)$$

from $\partial_\Delta\begin{pmatrix} 0 & \bar\Delta \\ \Delta & 0 \end{pmatrix}$

$=$ gap equation, now as function of temperature!

Matsubara sum

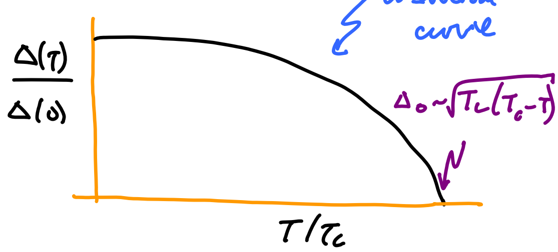
$$\frac{1}{g} = \frac{1}{L^d}\sum_{p}\frac{1 - 2n_F(\lambda_{\vec p})}{\lambda_{\vec p}} \qquad (4)$$

add cutoff

$$= \nu\int_0^{\omega_D} d\xi\, \frac{\tanh(\lambda(\xi)/2T)}{\lambda(\xi)} \qquad (5)$$

can be solved only numerically!

[see problem 6.7.2]



Ginzburg - Landau theory (set $\phi = \vec{A} = 0$ for now) $\boxed{\text{BCS 9}}$

Key idea: for $T \to T_c$, expand S (from 7.3) in powers of Δ!

$$\text{Define } \hat{\mathcal{G}}_0^{-1} \equiv \hat{\mathcal{G}}^{-1}\big|_{\Delta=0}, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix}, \text{ then} \qquad (1)$$

$$\mathrm{tr}\ln \hat{\mathcal{G}}^{-1} = \mathrm{tr}\ln\Big[\hat{\mathcal{G}}_0^{-1}(1+\hat{\mathcal{G}}_0\hat{\Delta})\Big] \qquad \to \begin{bmatrix} (\hat{\mathcal{G}}_0^{(p)})^{-1} & \Delta \\ \bar{\Delta} & (\hat{\mathcal{G}}_0^{(h)})^{-1} \end{bmatrix} \qquad (2)$$

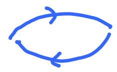
$$= \underbrace{\mathrm{tr}\ln \hat{\mathcal{G}}_0^{-1}}_{\text{free energy of noninteracting electron gas (boring)}} - \sum_{n=0}^{\infty} \frac{1}{2n}\,\mathrm{tr}\,(\hat{\mathcal{G}}_0\hat{\Delta})^{2n} \qquad (3)$$

[odd powers don't contribute to Z, because $\mathrm{tr}\begin{pmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{pmatrix} = 0$]

Consider $n=2$:

$$-\tfrac{1}{2}\,\mathrm{tr}\,(\hat{\mathcal{G}}_0\hat{\Delta})^2 = -\mathrm{tr}\Big[[\mathcal{G}_0]_{11}\,\Delta\,[\mathcal{G}_0]_{22}\,\bar{\Delta}\Big] \qquad (1)$$

(multiply out trace)

Δ  $\bar{\Delta}$ $\boxed{\text{BCS 10}}$

$$= -\sum_q \frac{T}{L^d}\sum_p \underbrace{[\hat{\mathcal{G}}_{0,p}]_{11}\,[\hat{\mathcal{G}}_{0,p-q}]_{22}}_{\mathcal{G}_p\,\mathcal{G}_{-p+q}} \;\Delta(q)\,\bar{\Delta}(q) \qquad (2)$$

$$\text{where we used} \quad \mathcal{G}_p^{(p)} \equiv \mathcal{G}_p, \quad \mathcal{G}_p^{(h)} \overset{(7.1)}{=} -\mathcal{G}_{-p}^{(p)} = -\mathcal{G}_{-p} \qquad (3)$$

Combine with 1st term of (7.3):

$$S^{(2)}[\Delta,\bar{\Delta}] = \sum_q |\Delta(q)|^2 \underbrace{\left(\frac{1}{g} - \frac{T}{L^d}\sum_p \mathcal{G}_p\,\mathcal{G}_{-p+q}\right)}_{=\ \Gamma_q^{-1}} \qquad (4)$$

$= \Gamma_q^{-1}$ = vertex function of (2.3) from Cooper channel!

RPA: $\langle V V \rangle$ described propagator of $\psi^\dagger\psi\,\psi^\dagger\psi$ interaction in direct channel

BCS: $\langle \bar{\Delta}\Delta \rangle$ " " of $\psi^\dagger\psi^\dagger\,\psi\psi$ interaction in Cooper channel.

BCS11

when $\Gamma_q^{-1} \to 0$, we get instability ($\int \mathcal{D}[\Delta,\bar\Delta]$ diverges),

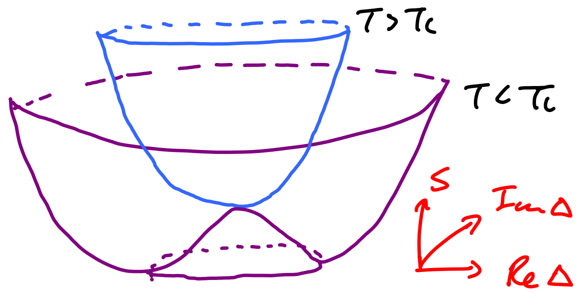
sign of action of $\Delta(q=0)$ mode changes;

assume $\Gamma_q^{-1} \sim T - T_c$ near T_c, then

$$S^{(2)}[\Delta,\bar\Delta] = \int d\tau \int d^d r \; \frac{r(T)}{2} |\Delta|^2 + O(\partial_r \Delta, \partial_\tau \Delta) \qquad (1)$$

↑ important for Goldstone modes!!

where $r(T) \sim (T - T_c)$.



For $T<T_c$, need also $|\Delta|^4$ contributions to stabilize path integral.

BCS12

For $\Delta = \Delta_0 =$ const, one may check: (for $n>1$)
(q = 0 modes)

$$S^{(2n)} \overset{(9.3)}{=} -\frac{1}{2n}\,\mathrm{tr}\left(\hat{G}_0 \hat{\Delta}\right)^{2n} \qquad (1)$$

as $T \to T_c$: small in $\left(\frac{\Delta}{T_c}\right)$

$$= -\frac{(-)^n}{2n} \sum_p \underbrace{\left(G_p G_{-p}\right)^n}_{\sum_{\omega_\ell} \int_{-\omega_D}^{\omega_D} d\xi \, \frac{1}{(\omega_\ell^2 + \xi_\ell^2)^n} \;\sim\; \sum_{\omega_\ell} \frac{1}{\omega_\ell^{2n-1}}} |\Delta|^{2n} \sim \text{const } \nu T \left(\frac{|\Delta|}{T}\right)^{2n} \qquad (2), (3)$$

Retain only $|\Delta|^4$ term:
(Δ/T is a small parameter near T_c!)

(2) $\sim \nu T^{-3}$, i.e. weak T-dependence near T_c

$$S[\Delta,\bar\Delta] = \int d\tau \int d^d r \left(\frac{r(T)}{2} \bar\Delta \Delta + \tilde{g} (\Delta\bar\Delta)^2 + O(\partial_r \Delta, \partial_\tau \Delta, \Delta^6) \right) \qquad (4)$$

if we include spatial gradients: $\quad + \frac{c(T)}{2} |\partial_r \Delta|^2 \qquad (5)$

"classical Landau-Ginzburg action"

↑ since τ-dependence was dropped (else: time-dependent LG)

- Above T_c: $\Delta = 0$ is only MF - solution (1) $\boxed{\text{BCSB}}$

- Below T_c: $\left.\frac{\delta S}{\delta \Delta}\right|_{\Delta=\Delta_0} = 0 \implies \bar{\Delta}_0 \left(r + 2\tilde{g} |\Delta_0|^2 \right) = 0$ (2)

$$\implies |\Delta_0| = \sqrt{\frac{-r}{2g}} \sim \sqrt{T_c (T_c - T)} \quad (3)$$

Broken symmetry: ground state has fixed phase,

e.g. $\Delta_0 \in \mathbb{R}$, , (4)

Goldstone modes: $\Delta = e^{2i\theta} \Delta_0$ explore deviations from fixed phase (5)

Difference to superfluid bosons: here we have <u>charged</u> particles, (6)
they couple to $(\phi, \vec{A})$, hence local $U(1)$ symmetry