Summary so far:

FRG-III.1

Flow eq. for generating funct.:

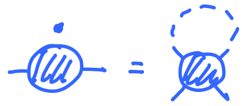
$$\dot\Gamma^\Lambda \overset{(II.11.2)}{=} -\,\mathrm{Tr}\left[\dot C^\Lambda\left(V^{\Lambda(1,1)}_{\bar\phi\phi} - \mathcal{G}^{0,\Lambda}\right)\right] \qquad (1)$$

$$V^{\Lambda(1,1)}_{\bar\phi\phi} \overset{(II.10.3)}{=} \left[1 - \mathcal{G}U + \left(\mathcal{G}U\mathcal{G}U + s\,\mathcal{G}\,\bar\delta^2\mathcal{G}^T\delta^2\right) + \left(\mathcal{G}U\mathcal{G}U\mathcal{G}U + \dots\right)\right]\mathcal{G} \qquad (2)$$

$$\Gamma \overset{(II.10.5)}{=} \sum_{m=0}^{\infty}\frac{s^m}{(m!)^2}\sum_{k'_1,\dots,k'_m}\;\sum_{k_1,\dots,k_m}\gamma_m(k'_1,\dots,k'_m;k_1,\dots,k_m)\,\bar\phi_{k'_1}\cdots\bar\phi_{k'_m}\,\phi_{k_m}\cdots\phi_{k_1} \qquad (3)$$

$$U_{q'q} \overset{(II.11.1)}{=} \sum_{m=1}^{\infty}\frac{s^m}{(m!)^2}\sum_{k'_1,\dots,k'_m}\;\sum_{k_1,\dots,k_m}\gamma_{m+1}(k'_1,\dots,k'_m,q';k_1,\dots,k_m,q)\,\bar\phi_{k'_1}\cdots\bar\phi_{k'_m}\,\phi_{k_m}\cdots\phi_{k_1} \qquad (4)$$

$$\dot\gamma_1(k',k) \overset{(II.12.3)}{=} \mathrm{Tr}\left[S\,\gamma_2(k',\cdot\,;k,\cdot)\right]$$



$$S \overset{(II.12.2)}{=} \mathcal{G}\,\dot C^\Lambda\,\mathcal{G} \quad \text{----} \qquad (5)$$

2-particle vertex function

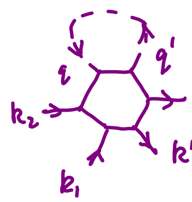
FRG-III.2

$$\dot\gamma_2(k'_1,k'_2,k_1,k_2) \overset{(I.13.1)}{=} \frac{\delta^4}{\delta\bar\phi_{k'_1}\,\delta\bar\phi_{k'_2}\,\delta\phi_{k_2}\,\delta\phi_{k_1}}\,\dot\Gamma\,\Big|_{\phi=0=\bar\phi} \qquad (1)$$

$$\overset{(1.1),(1.2)}{=} -\frac{\delta^4}{\delta\bar\phi_{k'_1}\,\delta\bar\phi_{k'_2}\,\delta\phi_{k_2}\,\delta\phi_{k_1}}\,\mathrm{Tr}\left[\dot C\left(-\mathcal{G}U + \mathcal{G}U\mathcal{G}U + s\,\mathcal{G}\,\bar\delta^2\Gamma\,\mathcal{G}^T\delta^2\Gamma\right)\mathcal{G}\right] \qquad (2)$$

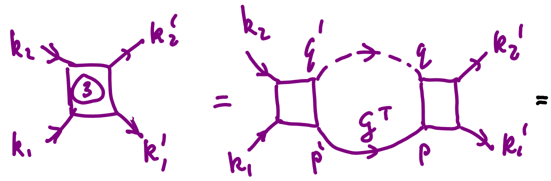
$$= -\frac{\delta^4}{\delta\bar\phi_{k'_1}\,\delta\bar\phi_{k'_2}\,\delta\phi_{k_2}\,\delta\phi_{k_1}}\,\mathrm{Tr}\Big[S\Big(\overset{1}{-U} + \overset{2}{U\mathcal{G}U} + \overset{3}{s\,\bar\delta^2\Gamma\,\mathcal{G}^T\delta^2\Gamma}\Big)\Big] \qquad (3)$$

$$1 = \mathrm{Tr}\,S\,\gamma_3(k'_1,k'_2,\cdot\,;k_1,k_2,\cdot)$$



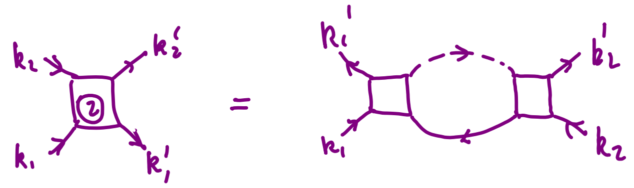
$$= m=3 \text{ vertex} \qquad (4)$$

$\boxed{\text{FRG III 3}}$



$$= -\mathrm{Tr}\Big[S \underbrace{\gamma_2(\cdot,\cdot;k_1,k_2)}_{\frac{\delta^2\Gamma}{\delta\bar\phi_{q'}\,\delta\bar\phi_{p'}}} G^T \underbrace{\gamma_2(k_1',k_2';\cdot,\cdot)}_{\frac{\delta^2\Gamma}{\delta\phi_q\,\delta\phi_p}} \Big]$$

$$= - S_{qq'}\, \gamma_2(q',p';k_1,k_2)\, G^T_{p'p}\, \gamma_2(k_1',k_2';p,q)$$

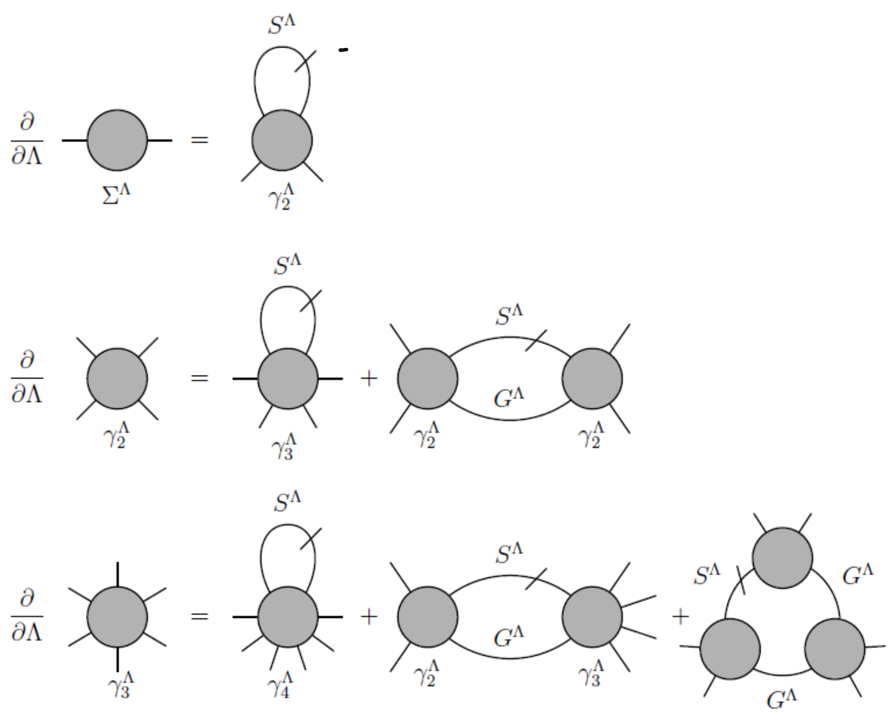


$$= -\mathrm{Tr}\big[S\, \gamma_2(k_1',\cdot;k_1,\cdot)\, G\, \gamma_2(k_2',\cdot;k_2,\cdot)$$

$$+ (k_1' \leftrightarrow k_2') \quad + (k_1 \leftrightarrow k_2) \quad + (k_1' \leftrightarrow k_2',\ k_1 \leftrightarrow k_2)$$

(modulo minus signs, which we did not keep track of here ...)

General structure of the hierarchy (without indices, ...) $\boxed{\text{FRG-III.4}}$



Truncation/Approximations

FRG III.5

$\gamma_m = 0$ for $m > 0$

γ_1, γ_2 independent of frequencies (but still energy conservation at vertices)

$$\frac{d}{d\Lambda}\gamma_2^\Lambda(k_1',k_2';k_1,k_2) =$$
$$\sum_{i\omega_n}\sum_{q,q',s,s'}\Big[-\mathcal{S}^\Lambda_{q,q'}(i\omega_n)\gamma_2^\Lambda(q',s';k_1,k_2)\mathcal{G}^\Lambda_{s,s'}(-i\omega_n)\gamma_2^\Lambda(k_1',k_2';s,q)$$
$$-\mathcal{S}^\Lambda_{q,q'}(i\omega_n)\gamma_2^\Lambda(k_1',q';k_1,s)\mathcal{G}^\Lambda_{s,s'}(i\omega_n)\gamma_2^\Lambda(k_2',s';k_2,q)$$
$$-\mathcal{S}^\Lambda_{q,q'}(i\omega_n)\gamma_2^\Lambda(k_2',q';k_2,s)\mathcal{G}^\Lambda_{s,s'}(i\omega_n)\gamma_2^\Lambda(k_1',s';k_1,q)$$
$$+\mathcal{S}^\Lambda_{q,q'}(i\omega_n)\gamma_2^\Lambda(k_2',q';k_1,s)\mathcal{G}^\Lambda_{s,s'}(i\omega_n)\gamma_2^\Lambda(k_1',s';k_2,q)$$
$$+\mathcal{S}^\Lambda_{q,q'}(i\omega_n)\gamma_2^\Lambda(k_1',q';k_2,s)\mathcal{G}^\Lambda_{s,s'}(i\omega_n)\gamma_2^\Lambda(k_1',s';k_2,q)\Big].$$

integrate ...

Toy example (to illustrate relation to perturbation theory)

FRG III.6

Choose generating functional $\quad W^c(\eta) \overset{(\mathrm{I}.3.5)}{=} \ln\Big[\Big\langle e^{-\frac{g}{4!}\varphi^4}\, e^{-\eta\varphi}\Big\rangle_0\Big]$ (1)

"interaction" "source term"

where $\langle F(\varphi)\rangle_0 \equiv \underbrace{\frac{1}{\sqrt{2\pi\, \mathcal{G}^0}}}_{\text{corresponds to } 1/Z_0}\int d\varphi\, e^{-\frac{\varphi^2}{2\mathcal{G}^0}} F(\varphi)$, $\quad \varphi, \eta, \mathcal{G}^0, g \in \mathbb{R}$ (2)

($= -\mathcal{G}^0$ of our theory so far ...) (2')

Since we do not distinguish φ from $\bar\varphi$, $\frac{\delta}{\delta\bar\varphi}\frac{\delta}{\delta\varphi} \to \frac{\delta^2}{\delta^2\varphi}$, hence (3)

we added extra $\frac{1}{2}$ in free action, so that [and $\frac{1}{(m!)^2} \to \frac{1}{(2m!)}$ later] (4)

$\langle\varphi\varphi\rangle_0 = \frac{d^2 W_0^c}{d\eta^2}\Big|_{\eta=0} = \mathcal{G}_0$, where $\langle F(\varphi)\rangle_0 \equiv$ (5)

[since $W_0^c = \ln\Big[\frac{1}{\sqrt{2\pi\,\mathcal{G}^0}}\int d\varphi\; e^{-\frac{1}{2\mathcal{G}_0}(\varphi+\mathcal{G}_0\eta)^2}\, e^{+\frac{\mathcal{G}_0\eta^2}{2}}\Big] = \frac{1}{2}\mathcal{G}_0\eta^2$] (6)

Exact integrals for correlators $\mathcal{G}^c_m$

To have a check on fRG results for vertex functions, let us obtain exact expressions for $\mathcal{G}^c_m$ in terms of the (numerically) known integrals

$$I_n = \frac{1}{\sqrt{2\pi \mathcal{G}^0}} \int d\varphi \; \varphi^n \, e^{-\frac{\varphi^2}{2\mathcal{G}^0}} \, e^{-\frac{g}{4!}\varphi^4} \qquad (= 0 \text{ for odd } n) \qquad (1)$$

$$W^c(\eta) \overset{(6.1)}{=} \ln\Big[\sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} I_n \Big] \qquad (2)$$

$$\mathcal{G}^c_0 = \ln I_0 \qquad (3)$$

$$\mathcal{G}^c_1 = \frac{d^2 W^c}{d\eta^2}\Big|_{\eta=0} = \frac{d}{d\eta}\Big(-\frac{1}{I_0} \sum_{n=1}^{\infty} \frac{n(-\eta)^{n-1}}{(n!)} I_n \Big)\Big|_{\eta=0} = \frac{I_2}{I_0} \qquad (4)$$

$$\mathcal{G}^c_2 = \frac{d^4 W^c}{d\eta^4} = \ldots = \frac{I_4}{I_0} - 3\frac{I_2^2}{I_0^2} \qquad (5)$$

Legendre-transformation to Γ:

$$\phi \overset{(I.11.1)}{=} -\frac{dW^c}{d\eta} = \sum_{n=1}^{\infty} \frac{(-\eta)^{n-1}}{(n-1)!} I_n \Big[\sum_{n=0}^{\infty} \frac{(-\eta)^n}{n!} I_n \Big]^{-1}, \Rightarrow \eta = \eta(\phi) \qquad (1)$$

$$\Gamma(\phi) \overset{(I.11.3)}{=} -W^c(\eta) - \phi\eta - \tfrac{1}{2} C \phi^2, \quad C = \mathcal{G}^{0^{-1}} \qquad (2)$$

$$\gamma_m = \frac{d^{2m}\Gamma}{d\phi^{2m}}\Big|_{\phi=0} \qquad (3)$$

To find relation between $\mathcal{G}^c_m$ and γ_m, proceed as on (I.12) ...

cancel

$$\frac{d\Gamma}{d\phi} \overset{(I.12.3)}{=} -\frac{dW^c}{d\eta}\frac{d\eta}{d\phi} - \eta - \phi\frac{d\eta}{d\phi} - C\phi = -\eta - C\phi \qquad (4)$$

$$\Rightarrow \quad \frac{d\eta}{d\phi} = -\frac{d^2\Gamma}{d\phi^2} - C, \qquad \frac{d\eta}{d\phi}\Big|_{\phi=0} = -\gamma_1 - C \qquad (5)$$

$$1 = \frac{d\phi}{d\phi} \overset{(8.1)}{=} -\frac{d}{d\phi}\frac{dW^c}{d\eta} = -\frac{d\eta}{d\phi}\frac{d^2W^c}{d\eta^2} \overset{(8.5)}{=} \left[\frac{d^2\Gamma}{d\phi^2} + c\right]\frac{d^2W^c}{d\eta^2} \qquad (1)$$

for $\phi = \eta = 0$: $\quad g_1^c = \frac{1}{c+\gamma_1} \quad \Rightarrow \quad \boxed{\gamma_1 = (g_1^c)^{-1} - c} \qquad (2)$

$$\Rightarrow \quad \frac{d\eta}{d\phi}\bigg|_{\phi=0} \overset{(8.5)}{=} -g_1^c \qquad (3)$$

term with $\frac{d^2\eta}{d\phi^2}\frac{d^3W^c}{d\eta^3} = 0$

$$0 = \frac{d^2}{d\phi^2}(1)\bigg|_{\phi=0} = \left[\underbrace{\frac{d^4\Gamma}{d\phi^4}}_{\gamma_2}\underbrace{\frac{d^2W^c}{d\eta^2}}_{g_1^c} + \underbrace{\left(\frac{d^2\Gamma}{d\phi^2}+c\right)}_{(g_1^c)^{-1}}\underbrace{\frac{d\eta}{d\phi}\frac{d\eta}{d\phi}}_{(g_1^c)^{-2}}\frac{d^4W^c}{d\eta^2}\right]_{\phi=0} \qquad (4)$$

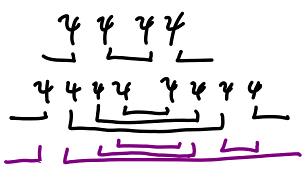
$$\boxed{\gamma_2 = -(g_1^c)^{-4}\, g^c} \qquad (5)$$

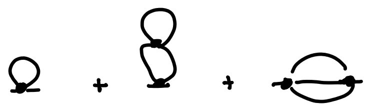
Hence, we can numerically find γ_2, γ_1, as functions of g, from g_4^c, g_2^c via (5), (3), (7.4-5)

Our goal is to check quality of various approximation schemes by comparing to (9.2), (9.5). Set $g^0 = 1$ henceforth.

Simple perturbation theory in g for $\langle e^{-\frac{g}{4!}\psi^4}\rangle$

use "Wick's theorem", "linked-cluster-theorem" (keep only connected diagrams) and keep only 1-particle irreducible diagrams:



$\Sigma = -\gamma_1 =$  $+ \dots$

$$= -\tfrac{1}{2}g + \tfrac{5}{12}g^2 + \mathcal{O}(g^4) \quad \text{(combinatorical factors)} \qquad (1)$$

$\gamma_2 =$ 

$$= g - \tfrac{3}{2}g^2 \qquad (2)$$

Self-consistent Hartree approximation

replace $$g\frac{\varphi^4}{4!} \to g\bar\varphi^2 \frac{\varphi^2}{2!} \,, \tag{1}$$

and require $$\bar\varphi^2 = \langle\varphi^2\rangle_{MF} = \frac{\int d\varphi\; \varphi^2\, e^{-\varphi^2(1+g\bar\varphi^2/2)}}{\int d\varphi\; \varphi^2 e^{-\varphi^2(1+g\bar\varphi^2/2)}} \tag{2}$$

$$= \frac{1}{1+g\bar\varphi^2/2} \tag{3}$$

solve: $$\bar\varphi^2 = \frac{2}{1+\sqrt{1+2g}} \tag{4}$$

$$\Sigma_{MF} = -\gamma_{1,MF} = -\tfrac{1}{2} g\,\bar\varphi^2 = -\frac{g}{1+\sqrt{1+2g}} \tag{5}$$

Flow equations

Choose $$G^{0,\Lambda} = \Lambda \,, \quad \text{and let } \Lambda \text{ flow from } \Lambda_0 = 0 \text{ to } \Lambda_{final} = 1 \tag{1}$$

$$[\,G^{0,\Lambda_0} = 0, \quad G^{0,\Lambda_{final}} = 1\,] \checkmark$$

$$\Rightarrow \quad C^\Lambda = \frac{1}{\Lambda} \,, \quad \dot C^\Lambda = -\frac{1}{\Lambda^2} \tag{2}$$

Take $$\gamma_m^{\Lambda_0} = \delta_{m,2}\, g \quad \text{i.e.} \quad \Gamma^{\Lambda_0 = 0} = \frac{1}{4!} g \varphi^4 \tag{3}$$

and $$G^\Lambda = \frac{1}{G_0^{-1,\Lambda} + \gamma_1^\Lambda} = \frac{1}{1/\Lambda + \gamma_1^\Lambda} = \frac{\Lambda}{1+\Lambda\gamma_1^\Lambda} \tag{4}$$

Flow eq: [as on p. II.8, with $G^{0,\Lambda} \xrightarrow{(6.2')} -G^{0,\Lambda}$, $C \longrightarrow -C$]

$$\dot\Gamma^\Lambda[\phi] \overset{(II.8.3)}{=} \tfrac{1}{2} \dot C^\Lambda \left(V^\Lambda_{\phi\phi} - G^{0,\Lambda} \right) \overset{(6)}{=} \tfrac{1}{2} \dot C^\Lambda \Big[-G^{0,\Lambda} + \underbrace{G^\Lambda (1+G^\Lambda U)^{-1}}_{G^\Lambda - G U G^\Lambda + GUGUG\ldots} \Big] \tag{5}$$

since $$V^\Lambda_{\phi\phi} \overset{(II.8.4)}{=} \left[\frac{d^2\Gamma}{d\phi^2} + C \right]^{-1} = \Big[\underbrace{\Big(\frac{d^2\Gamma}{d\phi^2} - \gamma_1^\Lambda \Big)}_{\overset{(II.9.1)}{\equiv} U[\phi]} + \underbrace{(\gamma_1^\Lambda + C)}_{\equiv G^{-1} =} \Big]^{-1} = \frac{1}{G^{-1} + U} \overset{(II.9.5)}{=} \frac{1}{1+GU}\, G \tag{6}$$

Expand: $$\Gamma^\Lambda(\phi) = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\, \gamma_m^\Lambda\, \phi^{2m} \qquad (1)$$

$$\mu^\Lambda(\phi) = \sum_{m=1}^{\infty} \frac{1}{(2m-2)!}\, \gamma_m^\Lambda\, \phi^{2m-2} - \gamma_1^\Lambda = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\, \gamma_{m+1}^\Lambda\, \phi^{2m} \qquad (2)$$

Insert into (12.6) and compare coefficients of ϕ:

$$\dot\gamma_0^\Lambda \overset{(12.5)}{=} \frac{1}{2}\dot C^\Lambda \left[-g^{0,\Lambda} + g^\Lambda \right] \overset{(12.2,4)}{=} \frac{1}{2}\left(-\frac{1}{\Lambda^2}\right)\left[-\Lambda + \frac{\Lambda}{1+\Lambda\gamma_1^\Lambda}\right] = \frac{1}{2}\, \frac{\gamma_1^\Lambda}{1+\Lambda\gamma_1^\Lambda} \qquad (3)$$

$$\dot\gamma_1^\Lambda = -\frac{1}{2}\dot C^\Lambda g u g = \frac{1}{2}\frac{1}{\Lambda^2}\left(\frac{\Lambda}{1+\Lambda\gamma_1^\Lambda}\right)^2 \gamma_2^\Lambda = \frac{1}{2}\, \frac{\gamma_2^\Lambda}{(1+\Lambda\gamma_1^\Lambda)^2} \qquad (4)$$

$$\dot\gamma_2^\Lambda = \frac{1}{2}\, \frac{\gamma_3^\Lambda}{(1+\Lambda\gamma_1^\Lambda)^2} - 3\Lambda \frac{(\gamma_2^\Lambda)^2}{(1+\Lambda\gamma_1^\Lambda)^3} \qquad (5)$$

(3)-(6) should be integrated, $\gamma_m = \int_0^1 d\Lambda\, \dot\gamma_m$, with $\gamma_2^{\Lambda_0} = g$

$$\dot\gamma_3^\Lambda = \frac{1}{2}\, \frac{\gamma_4^\Lambda}{(1+\Lambda\gamma_1^\Lambda)^2} - 15\Lambda \frac{\gamma_2^\Lambda\gamma_3^\Lambda}{(1+\Lambda\gamma_1^\Lambda)^3} + 45\Lambda^2 \frac{(\gamma_2^\Lambda)^3}{(1+\Lambda\gamma_1^\Lambda)^4} \qquad (6)$$

Consistency check: Recover pert. theory

Expand $$\gamma_m^\Lambda = \sum_{n=1}^{\infty} \gamma_{m,n}^\Lambda\, g^n \qquad (1)$$

due to n-particle irreducibility γ_m is at least of order g^m for $m \geq 3$

, insert into (13.3-6), compare powers of g:

Power g^1: $$\dot\gamma_{1,1}^\Lambda \overset{(13.4)}{=} \frac{1}{2}\gamma_{2,1}^\Lambda, \qquad \dot\gamma_{m,1}^\Lambda \overset{(13.5,6)}{=} 0 \quad \forall\, m \geq 2 \qquad (2)$$

Power g^2: $$\dot\gamma_{1,2}^\Lambda = \frac{1}{2}\, \frac{\gamma_2^\Lambda}{(1+\Lambda\gamma_1^\Lambda)^2}\Big|_{\text{order } g^2} = \frac{1}{2}\left[\gamma_{2,2}^\Lambda - \gamma_{2,1}^\Lambda \Lambda \gamma_{1,1}^\Lambda\right] \qquad (3)$$

$$\dot\gamma_{2,2}^\Lambda = -3\Lambda\, {\gamma_{2,1}^\Lambda}^2, \qquad \dot\gamma_{m,2}^\Lambda = 0 \quad \forall\, m \geq 3 \qquad (4)$$

Integrate, starting with $\gamma_{2,1}^{\Lambda_0 = 0} = 1$, $\gamma_{n,m}^{\Lambda_0 = 0} = 0$ $\forall$ other n,m. (5)

Start with eqs. of order g^1: $\boxed{\text{FRG II.15}}$

$$\dot\gamma_{2,1} \overset{(14.2)}{=} 0 \quad \Rightarrow \quad \gamma^\Lambda_{2,1} = 1 \quad \forall \Lambda \quad \Rightarrow \quad \gamma_2^\Lambda = g + \mathcal{O}(g^2) \qquad (1)$$

$$\gamma_{1,1} = \int_0^1 d\Lambda\, \dot\gamma_{1,1} \overset{(14.2)}{=} \frac12 \int_0^1 d\Lambda\, \gamma^\Lambda_{2,1} \overset{(1)}{=} \frac12 \Lambda \Rightarrow \gamma_1^\Lambda = \frac12 \Lambda g + \mathcal{O}(g^2) \qquad (2)$$

For $\Lambda = 1$, this reproduces (10.1), (10.2) ✓

$$\gamma_{2,2} \overset{(14.4)}{=} \int_0^1 d\Lambda \left(-3\Lambda \gamma^2_{2,1}\right) \overset{(1)}{=} \int_0^1 d\Lambda\,(-3\Lambda) = -\frac32 \Lambda^2 \qquad (3)$$

$$\gamma_{1,2} \overset{(14.3)}{=} \int_0^1 d\Lambda\, \frac12 \left[\gamma^\Lambda_{2,2} - \gamma^\Lambda_{2,1} \Lambda \gamma^\Lambda_{1,1}\right] \overset{(3),(2)}{=} \int_0^1 d\Lambda\, \frac12 \left[-\frac32 \Lambda^2 - 1\cdot\Lambda\cdot\frac12\Lambda\right] = -\frac23 \Lambda^3 \qquad (4)$$

$$\Rightarrow \quad \gamma_2 = g - \frac32 g^2 + \mathcal{O}(g^3) \qquad (5)$$

we recover perturbation theory order by order

$$\gamma_1 = \frac12 g - \frac23 g^2 + \mathcal{O}(g^3) \qquad (6)$$

When calculating γ_{m_c}: if we set $\gamma_{m_c+1} = 0$, $\boxed{\text{FRG-II.16}}$

equations close, and results are exact to order g^{m_c}. (1)

However, they are presumably <u>better</u> than pure pert. theory,

since we do <u>not</u>, in general expand $\gamma_m = \sum_{m,n} \gamma_{m,n} g^n$, but <u>integrate</u>.