Herleitung von $\frac{d^2\sigma}{dx\,dQ^2}$ aus $\frac{d^2\sigma}{d\Omega\,d\nu}$ ①

(I) Jacobi-Determinante $\frac{d^2\sigma}{d\Omega d\nu} \rightarrow \frac{d^2\sigma}{dx\,dQ^2}$

(i) $$\frac{d^2\sigma}{d\Omega d\nu} = \frac{d^3\sigma}{d\cos\vartheta\, d\varphi\, d\nu} \xrightarrow{d\varphi \to 2\pi} = \frac{d^2\sigma}{(2\pi\, d\cos\vartheta)\, d\nu}$$

(keine φ-Abhängigkeit des Wirkungsquerschnitts)

(ii) $\nu = Ey \rightsquigarrow y = \frac{\nu}{E}$

$x = \frac{Q^2}{2M\nu}$

$$Q^2 = 2\frac{EE'}{c^2}(1-\cos\vartheta) = 2M\nu x = 2MExy$$

$$\det\begin{bmatrix} \frac{\partial Q^2}{2\pi\partial\cos\vartheta} & \frac{\partial x}{2\pi\partial\cos\vartheta} \\ \frac{\partial Q^2}{\partial\nu} & \frac{\partial x}{\partial\nu}\end{bmatrix} = \det\begin{bmatrix} \frac{1}{2\pi}\cdot(-2EE'/c^2) & 0 \\ \frac{1}{2\pi}\cdot(2Mx) & -\frac{Q^2}{2M\nu^2}\end{bmatrix}$$

$$= \frac{1}{2\pi}(-2EE'/c^2)\cdot\left(-\frac{Q^2}{2M\nu^2}\right) - \frac{1}{2\pi}(2Mx)\cdot 0 = \frac{2}{2\pi}\cdot\frac{EE'}{c^2}\cdot\underbrace{\frac{Q^2}{2M\nu}}_{=x}\cdot\frac{1}{\nu}$$

$$\Rightarrow \det[\,..\,] = \frac{1}{\pi}\cdot\frac{EE'}{c^2}\cdot\frac{x}{\nu}$$

$$\Rightarrow \frac{d^2\sigma}{d\Omega d\nu} = \left(\frac{1}{\pi}\cdot\frac{EE'}{c^2}\frac{x}{\nu}\right)\frac{d^2\sigma}{dx\,dQ^2}$$

bzw. $$\frac{d^2\sigma}{dx\,dQ^2} = \frac{\pi\nu c^2}{EE'x}\,\frac{d^2\sigma}{d\Omega d\nu} \quad (*)$$

(II) $Q^2c^2 \approx 2EE'(1-\cos\vartheta) = 4EE'\sin^2\frac{\vartheta}{2}$

$\rightarrow$ (a) $\sin^2\frac{\vartheta}{2} = \frac{Q^2c^2}{4EE'}$

$\rightarrow$ (b) $\cos^2\frac{\vartheta}{2} = 1-\sin^2\frac{\vartheta}{2} = 1-\frac{Q^2c^2}{4EE'}$

(III) $$\frac{d^2\sigma}{d\Omega\, d\nu} = \left(\frac{d\sigma}{d\Omega}\right)_R \cdot \frac{1}{\nu}\cdot\left[F_2\cdot\cos^2\frac{\vartheta}{2} + \frac{2\nu}{Mc^2}F_1\cdot\sin^2\frac{\vartheta}{2}\right] \qquad \left(\begin{matrix}F_1 \equiv F_1(x,Q^2)\\ F_2 \equiv F_2(x,Q^2)\end{matrix}\right)$$

Rutherford

$$\overset{(a)(b)}{=} \left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{1}{\nu}\left[F_2\cdot\left(1-\frac{Q^2c^2}{4EE'}\right)+\frac{2\nu}{Mc^2}F_1\cdot\frac{Q^2c^2}{4EE'}\right]$$

$$\overset{\substack{Q^2=2MExy\\ E'=E(1-y)\\ \nu=yE}}{=} \left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{1}{\nu}\left[\left(\frac{1-y}{1-y}-\frac{Mxyc^2}{2E(1-y)}\right)F_2+\frac{2yE}{Mc^2}\cdot F_1\cdot\frac{Mxyc^2}{2E(1-y)}\right]$$

$$= \left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{1}{\nu(1-y)}\left[\left(1-y-\frac{Mxyc^2}{2E}\right)F_2+\frac{y^2}{2}\cdot 2xF_1\right]$$

$$\overset{\frac{E}{E'}=\frac{1}{1-y}}{=} \left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{E}{\nu E'}\left[\left(1-y-\frac{Mxyc^2}{2E}\right)F_2+\frac{y^2}{2}\cdot 2xF_1\right]$$

(I)(ii)⊛ $\Longrightarrow$ $$\frac{d^2\sigma}{dx\,dQ^2} = \left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{\pi\nu c^2}{EE'x}\cdot\frac{E}{\nu E'}\cdot\left[\text{——— '' ———}\right]$$

$$\Rightarrow \boxed{\frac{d^2\sigma}{dx\,dQ^2} = \pi\cdot\left(\frac{d\sigma}{d\Omega}\right)_R\cdot\frac{c^2}{E'^2}\cdot\left[\left(1-y-\frac{Mc^2xy}{2E}\right)\cdot\frac{F_2}{x}+\frac{y^2}{2}\frac{2xF_1}{x}\right]}$$

dabei ist $\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{4\alpha^2E'^2(\hbar c)^2}{(Qc)^4}$