Mathe-Vorkurs, Blatt 03: Integration

Lösung Hausaufgabe 1: Partielle Integration (*)

a) $\int dx\ x \ln x = \frac{1}{2}x^2 \ln x - \int dx\ \frac{1}{2}x^2 \frac{1}{x} = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int dx\ x = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

b) $\int dx\ x \cdot \sin x = -x \cos x + \int dx\ 1 \cdot \cos x = -x\cos x + \sin x + c$

c)
$$\int dx\ \sin x \cos x = sin^2 x - \left(\int dx\ \cos x \sin x\right) + c \quad \Big| + \int dx\ \cos x \sin x$$
$$\Rightarrow \int dx\ \sin x \cos x = \frac{1}{2}\sin^2 x + c$$

d)
$$\begin{aligned}
\int dx\ e^{ax}\cos(bx) &= \frac{1}{a}e^{ax}\cos(bx) + \int dx\ \frac{1}{a}e^{ax}\sin(bx)\cdot b = \\
&= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int dx\ \sin(bx)e^{ax} = \\
&= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\sin(bx)\frac{1}{a}e^{ax} - \int dx\ \cos(bx)\cdot\frac{b}{a}e^{ax}\right] \\
&= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) - \frac{b^2}{a^2}\int dx\ e^{ax}\cos(bx)
\end{aligned}$$
$$\iff \left(1+\frac{b^2}{a^2}\right)\int dx\ e^{ax}\cos(bx) = \frac{1}{a^2}e^{ax}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right)$$
$$\iff \int dx\ e^{ax}\cos(bx) = \frac{a^2 e^{ax}}{(a^2+b^2)a^2}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right)$$
$$\iff \int dx\ e^{ax}\cos(bx) = \frac{e^{ax}}{(a^2+b^2)}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right) + c$$

e)
$$\begin{aligned}
\int_0^1 dx\ xe^{-2x} &= \left[x\left(-\frac{1}{2}\right)e^{-2x}\right]_0^1 - \int_0^1 dx\ 1\left(-\frac{1}{2}\right)e^{-2x} = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 dx\ e^{-2x} \\
&= \left[-\frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x}\right]_0^1 = \left[-\frac{1}{2}e^{-2x}\left(x+\frac{1}{2}\right)\right]_0^1 \\
&= \left(-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right) + \frac{1}{4} = \frac{1}{4} - \frac{3}{4}e^{-2}
\end{aligned}$$

f)
$$\begin{aligned}
\int_1^2 dx\ x^2\cdot\ln x &= \left[\frac{1}{3}x^3\ln x\right]_1^2 - \int_1^2 dx\ \frac{1}{x}\cdot x^3 = \left[\frac{1}{3}x^3\ln x\right]_1^2 - \frac{1}{3}\int_1^2 dx\ x^2 = \left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right]_1^2 \\
&= \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{3}\ln 1 + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{9}
\end{aligned}$$

g) $\int_0^1 dx\ (1+x)e^x = [(1+x)e^x]_0^1 - \int_0^1 dx\ 1\cdot e^x = [(1+x)e^x - e^x]_0^1 = [xe^x]_0^1 = e$

h) $$\int_1^2 dx\ \frac{1}{x^2}\ln x = \int_1^2 dx\ x^{-2}\ln x = \left[\ln x\left(-\frac{1}{x}\right)\right]_1^2 + \int_1^2 dx\ \frac{1}{x}\cdot\frac{1}{x} = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 dx\ \frac{1}{x^2} =$$
$$-\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^2 = \left[-\frac{lnx - 1}{x}\right]_1^2 = \frac{-\ln 2 - 1}{2} = \frac{-1}{1} = \frac{-\ln 2 + 1}{2} = \frac{1}{2} - \frac{\ln 2}{2}$$

Lösung Hausaufgabe 2: Partialbruchzerlegung (*)

a)
$$\frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2} =$$
$$= \frac{Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + C + 2C}{(x+2)(x-1)^2}$$
$$= \frac{x^2(A+B) + x(2A+B+C) + A - 2B + 2C)}{(x+2)(x-1)^2}$$
$$\frac{4x-1}{(x+2)(x-1)^2} \overset{!}{=} \frac{x^2(A+B) + x(-2A+B+C) + (A-2B+2C)}{(x+2)(x-1)^2}$$

(i)	$A + B = 0$	$A = -B \downarrow$	$A = -1$
(ii)	$-2A + B + C = 4$	$C = 4 - 3B \downarrow$	$C = 1 \uparrow$
(iii)	$A - 2B + 2C = -1$	$-3B + 8 - 6B = -1 \Rightarrow$	$B = 1 \uparrow$

$$\int dx\ \frac{4x-1}{(x+2)(x-1)^2} = -\int dx\ \frac{1}{x+2} + \int dx\ \frac{1}{x-1} + \int dx\ \frac{1}{(x-1)^2}$$
$$= -\ln|x+2| + \ln x - 1 - \frac{1}{x-1} + c$$

b) analog eben bringt: Nenner hat Zerlegung $N = (x-3)(x-1)(x+1)$

(i)	$A + B + C = 0$	$A = -B - C \downarrow$	$A = \frac{5}{8}$
(ii)	$-2C - 4B = 1$	$C = -\frac{1}{2} - 2B \downarrow$	$C = -\frac{3}{4} \uparrow$
(iii)	$A - 3B + 3C = 2$	$B + C - 3B - 3C = 2 \Rightarrow$	$B = \frac{1}{8} \uparrow$

$$\int dx\ \frac{x+2}{(x+1)(x-1)(x-3)} = \int dx\ \frac{5}{8(x-3)} - \int dx\ \frac{3}{4(x-1)} + \int dx\ \frac{1}{8(x+1)}$$
$$= \frac{5}{8}\ln|x-3| - \frac{3}{4}\ln|x-1| + \frac{1}{8}\ln|x+1|$$

c) analog:

$$\int dx\ \frac{x^2 + 11\cdot x - 36}{x^3 + 5\cdot x^2 - 13\cdot x + 7} = \int dx\ \frac{-3}{(x-1)^2} + \int dx\ \frac{2}{x-1} - \int dx\ \frac{1}{x+7}$$
$$= \frac{3}{x-1} + 2\ln|x-1| - \ln|x+7|$$

d) $$\int dx\ \frac{7x^2-36x+21}{(x-1)^2(x^2-9)} = \int dx\ \frac{3}{x-1} - \int dx\ \frac{2}{x+3} + \int dx\ \frac{1}{(x-1)^2} - \int dx\ \frac{1}{x-3}$$
$$= 3\ln|x-1| - 2\ln|x+3| - \frac{1}{x-1} - \ln|x-3|$$

Lösung Hausaufgabe 3: Substitutionsmethode (*)

a) $\int dx\ (5x-4)^3 \overset{y=5x-4}{\underset{dy=5dx}{=}} \frac{1}{5}\int dy\ y^3 = \frac{1}{20}[y^4] = \frac{1}{20}(5x-4)^4$

b) $\int dx\ x^2\sqrt{2x^3+4} \overset{y=2x^3+4}{\underset{dy=6x^2dy}{=}} \frac{1}{6}\int dy\ \sqrt{y} = \frac{1}{6}\frac{2}{3}[\sqrt{y^3}] = \frac{1}{9}(2x^3+4)^{\frac{3}{2}}$

c) $\int dx\ \frac{1}{\sqrt{7-3x}} \overset{y=7-3x}{\underset{dy=-3dx}{=}} -\frac{1}{3}\int dy\ \frac{1}{\sqrt{y}}dy = -\frac{2}{3}[\sqrt{y}] = -\frac{2}{3}\sqrt{7-3x}$

d) $\int dx\ (x^2+8)^{10}\cdot x \overset{y=x^2+8}{\underset{dy=2xdy}{=}} \frac{1}{2}\int dy\ y^{10} = \frac{1}{22}[y^{11}] = \frac{(x^2+8)^{11}}{22}$

e) $\int dx\ \cos^4 x\sin x \overset{y=\cos x}{\underset{dy=-\sin x dx}{=}} -\int dy\ y^4 = -\frac{1}{5}[y^5] = -\frac{\cos x^5}{5}$

f) $\int dx\ \frac{2x+3}{x^2+3x+5} \overset{y=x^2+3x+5}{\underset{dy=(2x+3)dx}{=}} \int dy\ \frac{1}{y} = [\ln y] = \ln(x^2+3x+5)$

g) $\int dx\ xe^{x^2} \overset{y=x^2}{\underset{dy=2xdx}{=}} \frac{1}{2}\int dy\ e^y = \frac{1}{2}[e^y] = \frac{1}{2}e^{x^2}$

h) $\int_0^1 dx\ \frac{x}{x^2+1} \overset{y=x^2+1}{\underset{dy=2xdx}{=}} \frac{1}{2}\int_1^2 \frac{dy}{y} = \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \frac{1}{2}\ln 2$

i) $\int_0^{100} dx\ e^{\sqrt{x}} \overset{y=\sqrt{x}}{\underset{dy=\frac{1}{\sqrt{x}}dx}{=}} \int_0^{10} dy\ 2ye^y = 2[e^y y]_0^{10} - 2\int_0^{10} dy\ e^y = 2e^y(y-1)\Big|_0^{10} = 18e^{10}+2$