Lösungen zum Übungsblatt 3

Aufgabe 1 (Partielle Integration).

a) $\int x \ln x \,\mathrm{d}\,x = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x} \,\mathrm{d}\,x = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \,\mathrm{d}\,x = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

b) $\int x \cdot \sin x \,\mathrm{d}\,x = -x\cos x + \int 1 \cdot \cos x \,\mathrm{d}\,x = -x\cos x + \sin x + c$

c)
$$\int \sin x \cos x \,\mathrm{d}\,x = sin^2 x - \int \cos x \sin x \,\mathrm{d}\,x + c \quad \Big| + \int \cos x \sin x \,\mathrm{d}\,x$$
$$\Rightarrow \int \sin x \cos x \,\mathrm{d}\,x = \frac{1}{2}\sin^2 x + c$$

d)
$$\begin{aligned}\int e^{ax}\cos(bx)\,\mathrm{d}\,x &= \frac{1}{a}e^{ax}\cos(bx) + \int \frac{1}{a}e^{ax}\sin(bx)\cdot b\,\mathrm{d}\,x = \\ &= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int \sin(bx)e^{ax}\,\mathrm{d}\,x = \\ &= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\sin(bx)\frac{1}{a}e^{ax} - \int \cos(bx)\cdot\frac{b}{a}e^{ax}\,\mathrm{d}\,x\right] = \\ &= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx) - \frac{b^2}{a^2}\int e^{ax}\cos(bx)\,\mathrm{d}\,x\end{aligned}$$
$$\iff \left(1+\frac{b^2}{a^2}\right)\int e^{ax}\cos(bx)\,\mathrm{d}\,x = \frac{1}{a^2}e^{ax}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right)$$
$$\iff \int e^{ax}\cos(bx)\,\mathrm{d}\,x = \frac{a^2e^{ax}}{(a^2+b^2)a^2}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right)$$
$$\iff \int e^{ax}\cos(bx)\,\mathrm{d}\,x = \frac{e^{ax}}{(a^2+b^2)}\left(\cos(bx)\cdot a + b\cdot\sin(bx)\right) + c$$

e)
$$\int_0^1 xe^{-2x}\,\mathrm{d}\,x = \left[x\left(-\frac{1}{2}\right)e^{-2x}\right]_0^1 - \int_0^1 1\left(-\frac{1}{2}\right)e^{-2x}\,\mathrm{d}\,x = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\,\mathrm{d}\,x$$
$$= \left[-\frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x}\right]_0^1 = \left[-\frac{1}{2}e^{-2x}\left(x+\frac{1}{2}\right)\right]_0^1 = \left(-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right) + \frac{1}{4} = \frac{1}{4} - \frac{3}{2}e^{-2}$$

f)
$$\int_1^2 x^2\cdot\ln x\,\mathrm{d}\,x = \left[\frac{1}{3}x^3\ln x\right]_1^2 - \int_1^2 \frac{1}{x}\cdot x^3\,\mathrm{d}\,x = \left[\frac{1}{3}x^3\ln x\right]_1^2 - \frac{1}{3}\int_1^2 x^2\,\mathrm{d}\,x = \left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right]_1^2$$
$$= \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{3}\ln 1 + \frac{1}{9} = \frac{8}{3}\ln 2 - \frac{7}{9}$$

g) $\int_0^1 (1+x)e^x\,\mathrm{d}\,x = [(1+x)e^x]_0^1 - \int_0^1 1\cdot e^x\,\mathrm{d}\,x = [(1+x)e^x - e^x]_0^1 = [xe^x]_0^1 = e$

h)
$$\int_1^2 \frac{1}{x^2}\ln x\,\mathrm{d}\,x = \int_1^2 x^{-2}\ln x\,\mathrm{d}\,x = \left[\ln x\left(-\frac{1}{x}\right)\right]_1^2 + \int_1^2 \frac{1}{x}\cdot\frac{1}{x}\,\mathrm{d}\,x = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 \frac{1}{x^2}\,\mathrm{d}\,x =$$
$$-\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^2 = \left[-\frac{lnx - 1}{x}\right]_1^2 = \frac{-\ln 2 - 1}{2} = \frac{-1}{1} = \frac{-\ln 2 + 1}{2} = \frac{1}{2} - \frac{\ln 2}{2}$$

Aufgabe 2 (Partialbruchzerlegung).

a)

$$\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}=$$
$$=\frac{Ax^2-2Ax+A+Bx^2+Bx-2B+C+2C}{(x+2)(x-1)^2}$$
$$=\frac{x^2(A+B)+x(2A+B+C)+A-2B+2C)}{(x+2)(x-1)^2}$$
$$\frac{4x-1}{(x+2)(x-1)^2}\overset{!}{=}\frac{x^2(A+B)+x(-2A+B+C)+(A-2B+22C)}{(x+2)(x-1)^2}$$

$$\begin{array}{llll} (i) & A+B=0 & A=-B\downarrow & A=-1 \\ (ii) & -2A+B+C=4 & C=4-3B\downarrow & C=1\uparrow \\ (iii) & A-2B+2C=-1 & -3B+8-6B=-1\Rightarrow & B=1\uparrow \end{array}$$

$$\int\frac{4x-1}{(x+2)(x-1)^2}\,\mathrm{d}\,x=-\int\frac{1}{x+2}\,\mathrm{d}\,x+\int\frac{1}{x-1}\,\mathrm{d}\,x+\int\frac{1}{(x-1)^2}\,\mathrm{d}\,x=-\ln|x+2|+\ln x-1-\frac{1}{x-1}+c$$

b) analog eben bringt: Nenner hat Zerlegung $N=(x-3)(x-1)(x+1)$

$$\begin{array}{llll} (i) & A+B+C=0 & A=-B-C\downarrow & A=\frac{5}{8} \\ (ii) & -2C-4B=1 & C=-\frac{1}{2}-2B\downarrow & C=-\frac{3}{4}\uparrow \\ (iii) & A-3B+3C=2 & B+C-3B-3C=2\Rightarrow & B=\frac{1}{8}\uparrow \end{array}$$

$$\int\frac{x+2}{(x+1)(x-1)(x-3)}\,\mathrm{d}\,x=\int\frac{5}{8(x-3)}\,\mathrm{d}\,x-\int\frac{3}{4(x-1)}\,\mathrm{d}\,x+\int\frac{1}{8(x+1)}\,\mathrm{d}\,x$$
$$=\frac{5}{8}\ln|x-3|-\frac{3}{4}\ln|x-1|+\frac{1}{8}\ln|x+1|$$

c) analog:

$$\int\frac{x^2+11\cdot x-36}{x^3+5\cdot x^2-13\cdot x+7}\,\mathrm{d}\,x=\int\frac{-3}{(x-1)^2}\,\mathrm{d}\,x+\int\frac{2}{x-1}\,\mathrm{d}\,x-\int\frac{1}{x+7}\,\mathrm{d}\,x$$
$$=\frac{3}{x-1}+2\ln|x-1|-\ln|x+7|$$

d) $$\int\frac{7x^2-36x+21}{(x-1)^2(x^2-9)}\,\mathrm{d}\,x=\int\frac{3}{x-1}\,\mathrm{d}\,x-\int\frac{2}{x+3}\,\mathrm{d}\,x+\int\frac{1}{(x-1)^2}\,\mathrm{d}\,x-\int\frac{1}{x-3}\,\mathrm{d}\,x$$
$$=3\ln|x-1|-2\ln|x+3|-\frac{1}{x-1}-\ln|x-3|$$

Aufgabe 3 (Substitutionsmedthode).

a) $\int (5x-4)^3 \,\mathrm{d}\,x \overset{y=5x-4}{\underset{\mathrm{d}\,y=5\,\mathrm{d}\,x}{=}} \frac{1}{5}\int y^3 \,\mathrm{d}\,y = \frac{1}{20}[y^4] = \frac{1}{20}(5x-4)^4$

b) $\int x^2\sqrt{2x^3+4}\,\mathrm{d}\,x \overset{y=2x^3+4}{\underset{\mathrm{d}\,y=6x^2\,\mathrm{d}\,y}{=}} \frac{1}{6}\int \sqrt{y}\,\mathrm{d}\,y = \frac{1}{6}\frac{2}{3}[\sqrt{y^3}] = \frac{1}{9}(2z^3+4)^{\frac{3}{2}}$

c) $\int \frac{1}{\sqrt{y-3x}}\,\mathrm{d}\,x \overset{y=7-3x}{\underset{\mathrm{d}\,y=-3\,\mathrm{d}\,x}{=}} -\frac{1}{3}\int \frac{1}{\sqrt{y}}\,\mathrm{d}\,y = -\frac{2}{3}[\sqrt{y}] = -\frac{2}{3}\sqrt{7-3x}$

d) $\int (x^2+8)^{10}\cdot x\,\mathrm{d}\,x \overset{y=x^2+8}{\underset{\mathrm{d}\,y=2x\,\mathrm{d}\,y}{=}} \frac{1}{2}\int y^{10}\,\mathrm{d}\,y = \frac{1}{22}[y^{11}] = \frac{(x^2+8)^{11}}{22}$

e) $\int \cos^4 x \sin x\,\mathrm{d}\,x \overset{y=\cos x}{\underset{\mathrm{d}\,y=-\sin x\,\mathrm{d}\,x}{=}} -\int y^4\,\mathrm{d}\,y = -\frac{1}{5}[y^5] = -\frac{\cos x^5}{5}$

f) $\int \frac{2x+3}{x^2+3x+5}\,\mathrm{d}\,x \overset{y=x^2+3x+5}{\underset{\mathrm{d}\,y=(2x+3)\,\mathrm{d}\,x}{=}} \int \frac{1}{y}\,\mathrm{d}\,y = [\ln y] = \ln(x^2+3x+5)$

g) $\int xe^{x^2}\,\mathrm{d}\,x \overset{y=x^2}{\underset{\mathrm{d}\,y=2x\,\mathrm{d}\,x}{=}} \frac{1}{2}\int e^y\,\mathrm{d}\,y = \frac{1}{2}[e^y] = \frac{1}{2}e^{x^2}$

h) $\int_0^1 \frac{x}{x^2+1}\,\mathrm{d}\,x \overset{y=x^2+1}{\underset{\mathrm{d}\,y=2x\,\mathrm{d}\,x}{=}} \frac{1}{2}\int_1^2 \frac{\mathrm{d}\,y}{y} = \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \frac{1}{2}\ln 2$

i) $\int_0^{100} e^{\sqrt{x}}\,\mathrm{d}\,x \overset{y=\sqrt{x}}{\underset{\mathrm{d}\,y=\frac{1}{\sqrt{x}}\,\mathrm{d}\,x}{=}} \int_0^{10} 2ye^y\,\mathrm{d}\,y = 2[e^y y]_0^{10} - 2\int_0^{10} e^y\,\mathrm{d}\,y = 2e^y(y-1)\Big|_0^{10} = 18e^{10}$