Vorlesung IV

Ableitung spezieller Funktionen (Fortsetzung):

$$u(x) = \sin(x), \qquad u'(x) = \cos(x) \qquad (1)$$

$$v(x) = \cos(x), \qquad v'(x) = -\sin(x) \qquad (2)$$

Bsp. 1

$$t(x) \equiv \tan(x) \qquad t'(x) = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$= \frac{\sin(x)}{\cos(x)}, \qquad \sec(x) \equiv \frac{1}{\cos(x)} = \text{"secans"}$$

Quotientenregel

$$t'(x) = \frac{\cos \cdot \sin' - \sin \cos'}{\cos^2(x)} = \frac{\overbrace{\cos \cdot \cos - \sin(-\sin)}^{=1}}{\cos^2(x)} = \frac{1}{\cos^2(x)} \qquad (3)$$

Bsp. 2 $\quad \cot(x) \equiv \dfrac{\cos(x)}{\sin(x)} = \text{"cotangens"}, \quad \dfrac{1}{\sin(x)} \equiv \operatorname{cosec}(x) \qquad (1)$

$$\cot'(x) = \frac{\sin(-\sin) - \cos(\cos)}{\sin^2} = \frac{-1}{\sin^2} = -\operatorname{cosec}^2(x) \qquad (2)$$

Bsp. 3 $\quad \dfrac{d}{dx}\arctan(x) = \dfrac{1}{x^2+1} \qquad (3)$

Beweisidee: Ausgangspunkt: $\dfrac{d}{dx}\arctan\left(\tan(x)\right) = \dfrac{d}{dx}x = 1 \qquad (4)$

Kettenregel: $\quad \arctan'\left(\tan(x)\right) \cdot \underbrace{\tan'(x)}_{(1.3)\ = \frac{1}{\cos^2 x}} = 1 \qquad (5)$

(5) $\cdot \cos^2 x$: $\quad \arctan'(\tan(x)) = \cos^2 x \qquad (6)$

Substitution (führe neue Variable ein): $\tan(x) = y \qquad (7)$

③

Nebenrechnung:

$$y = \tan(x) \qquad (1)$$

$$y^2 = \tan^2 = \frac{\sin^2}{\cos^2} = \frac{1-\cos^2}{\cos^2} = \frac{1}{\cos^2} - 1 \qquad (2)$$

$$\frac{1}{\cos^2(x)} = 1 + y^2 \qquad (3)$$

$$\cos^2(x) = \frac{1}{1+y^2} \qquad (4)$$

(2.6):

$$\arctan'(y) = \left(\cos^2(x)\right) \overset{(4)}{=} \frac{1}{1+y^2} = \frac{1}{1+\tan^2(x)}$$

Endergebnis:

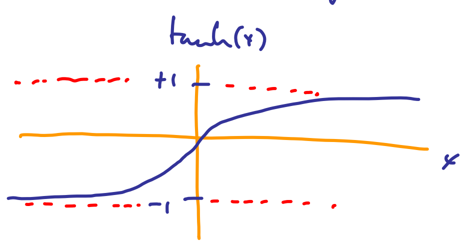
$$\arctan'(y) = \frac{1}{1+y^2}$$

④

Bsp. 3:

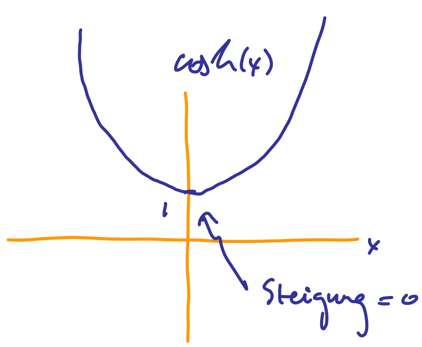
$$f(x) = \tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad (1)$$

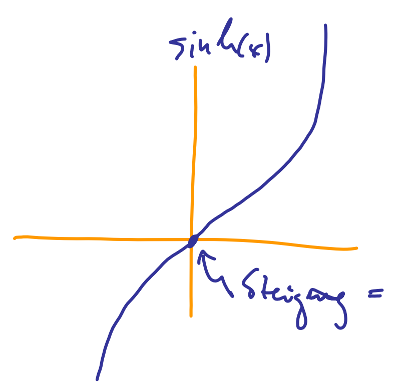
"tangens hyperbolicus"



$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2} \qquad (2)$$

$$\cosh(x) \equiv \frac{e^x + e^{-x}}{2} \qquad (3)$$





$$\sinh'(x) \overset{(4.2)}{\underset{(KR)}{=}} \frac{1}{2}\left(e^{x} - e^{-x}\cdot(-1)\right) \overset{(4.3)}{=} \cosh(x) \quad (1)$$

$$\cosh'(x) \overset{(4.3)}{=} \frac{1}{2}\left(e^{x} + e^{-x}(-1)\right) \overset{(4.2)}{=} \sinh(x) \quad (2)$$

⑤

sehr analog!

Vergleiche:

$$\sin'(x) = \cos(x) \quad (3)$$

$$\cos'(x) = -\sin(x) \quad (4)$$

Grund:

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad (5)$$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad (6)$$

Also gilt:

$$\sin(x) = \frac{1}{i}\sinh(ix)$$

$$\cos(x) = \cosh(ix)$$

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⑥

$$\tanh'(x) = \frac{d}{dx}\,\frac{\sinh(x)}{\cosh(x)} = \frac{\cosh\cdot\cosh - \sinh\cdot\sinh}{\cosh^2} \quad (1)$$

$$= \frac{1}{\cosh^2(x)} \equiv \operatorname{sech}^2(x) \quad (2) \qquad \frac{1}{\cosh(x)} \equiv \operatorname{sech}(x) \quad (3)$$

→ "secans hyperbolicus"

Nebenrechnung:

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}\left[\left(e^{x}+e^{-x}\right)^2 - \left(e^{x}-e^{-x}\right)^2\right]$$

$$= \frac{1}{4}\left[\left(\cancel{e^{2x}} + 2\cdot\underbrace{e^{x}\cdot e^{-x}}_{=1} + \cancel{e^{-2x}}\right) - \left(\cancel{e^{2x}} - 2\underbrace{e^{x}\cdot e^{-x}}_{=1} + \cancel{e^{-2x}}\right)\right]$$

$$= \frac{1}{4}\left[2+2\right] = 1 = \cosh^2(x) - \sinh^2(x) \quad (4)$$

(4) ist analogon zu $\cos^2(x) + \sin^2(x) = 1.$ (5)

Bsp 4: Lange Kettenregel:

$$f(x) = \sin^2\left(\sqrt{x^2+1}\right) \qquad (1)$$

$$f'(x) \overset{(7)}{=} 2\sin\left(\sqrt{x^2+1}\right)\cdot\cos\left(\sqrt{x^2+1}\right)\cdot \qquad (2)$$

$$\frac{1}{2}\,\frac{1}{\sqrt{x^2+1}}\cdot 2x\,, \qquad (3)$$

$$\overset{(5)}{=} \sin\left(2\sqrt{x^2+1}\right)\cdot\frac{x}{\sqrt{x^2+1}} \qquad (4)$$

$P(x_1) = x_1^2$

$Q(x_2) = \sin x_2$

$R(x_3) = \sqrt{x_3}$

$S(x_4) = x_4+1$

$T(x_5) = x_5^2$

Identität: $2\sin y\cdot\cos y = \sin(2y)$ (5)

$$f(x) = P\Big(Q\big(R\big(S\big(T(x)\big)\big)\big)\Big) \qquad (6)$$

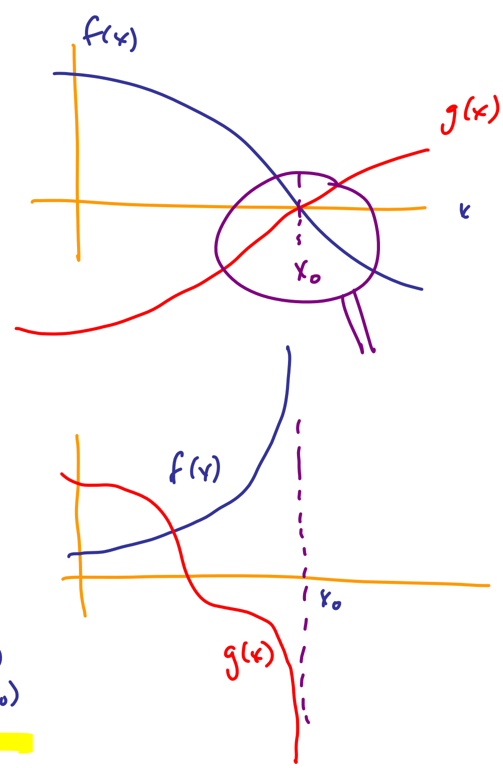
$$f'(x) = P'(Q)\,Q'(R)\,R'(S)\,S'(T)\,T'(x) \qquad (7)$$

3.5 Regel v. Bernoulli - L'Hopital

$$L = \lim_{x\to x_0}\frac{f(x)}{g(x)} = ?$$

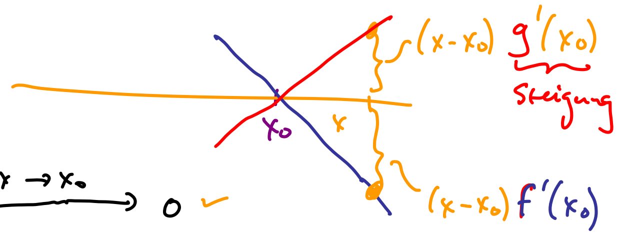
Falls $f(x)$ und $g(x)$ beide bei x_0 den Grenzwert 0, oder beide den Grenzwert $\pm\infty$ haben, aber in einer Umgebung v. x_0 differenzierbar sind, gilt:

$$L = \lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}$$



Beweisidee: (für $f(x_0) = g(x_0) = 0$):

In der Nähe von x_0 gilt:



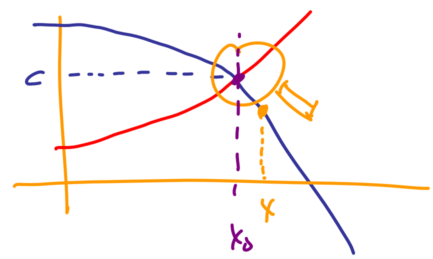
$$\left.\begin{aligned} f(x) &\simeq (x-x_0)\, f'(x_0) \\ g(x) &\simeq (x-x_0)\, g'(x_0) \end{aligned}\right\} \xrightarrow{\text{für } x \to x_0} 0 \checkmark$$

↳ "Linearisieren der Funktion"

$$L = \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\cancel{(x-x_0)}\, f'(x_0)}{\cancel{(x-x_0)}\, g'(x_0)} = \frac{f'(x_0)}{g'(x_0)}$$

Warum funktioniert das nur falls $\frac{f(x_0)}{g(x_0)} = \frac{0}{\text{oder}}\ \pm\infty$?

Bsp: Falls $f(x_0) = g(x_0) = c$:



Dann liefert Linearisierung:

$$f(x) = c + (x-x_0)\, f'(x_0)$$

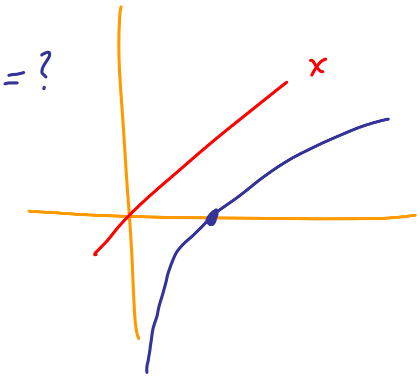
$$g(x) = c + (x-x_0)\, g'(x_0)$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{c + \cancel{(x-x_0)}\, f'(x_0)}{c + \cancel{(x-x_0)}\, g'(x_0)} = \frac{c}{c} = 1$$

Was ist, wenn $x_0 \to \infty$?

Versuche Substitution: $y = 1/x$, $y_0 = 0$

$$\lim_{x \to 0} \left(x \ln(x) \right) \text{ "=" } 0 \cdot \infty = ?$$



L'Hopital:

Umformung: $x \cdot \ln(x) = \dfrac{\ln(x)}{(1/x)}$

$$\lim_{x \to \infty} \left(\frac{\ln x}{x^n} \right) = 0 .$$