Multiplikation komplexer Zahlen in Polardarstellung

$$z_1 = a_1 + i b_1 = \underbrace{|z_1|}_{r_1}\left(\underbrace{\cos\varphi_1}_{c_1} + i\,\underbrace{\sin\varphi_1}_{s_1}\right) = r_1(c_1 + i s_1)$$

$$z_2 = a_2 + i b_2 = \underbrace{|z_2|}_{r_2}\left(\underbrace{\cos\varphi_2}_{c_2} + i\,\underbrace{\sin\varphi_2}_{s_2}\right) = r_2(c_2 + i s_2)$$

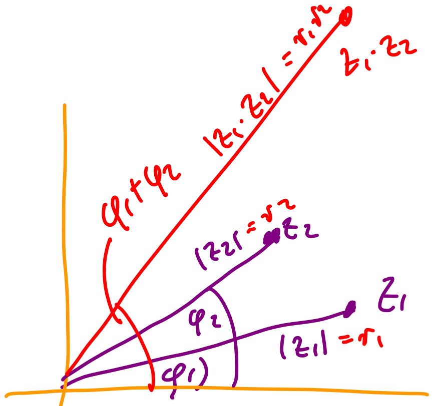
$$z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

$$= r_1(c_1 + i s_1) \cdot r_2(c_2 + i s_2)$$

$$= r_1 \cdot r_2 \left[(c_1 c_2 - s_1 \cdot s_2) + i(c_1 s_2 + s_1 c_2)\right]$$

$$= r_1 r_2 \left[\underbrace{(\cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2)}_{\cos(\varphi_1+\varphi_2)} + i\underbrace{(\cos\varphi_1 \sin\varphi_2 + \sin\varphi_1 \cos\varphi_2)}_{\sin(\varphi_1+\varphi_2)}\right]$$

$$= (r_1 r_2)\left[\cos(\varphi_1+\varphi_2) + i \sin(\varphi_1+\varphi_2)\right]$$



Einheitskreis:

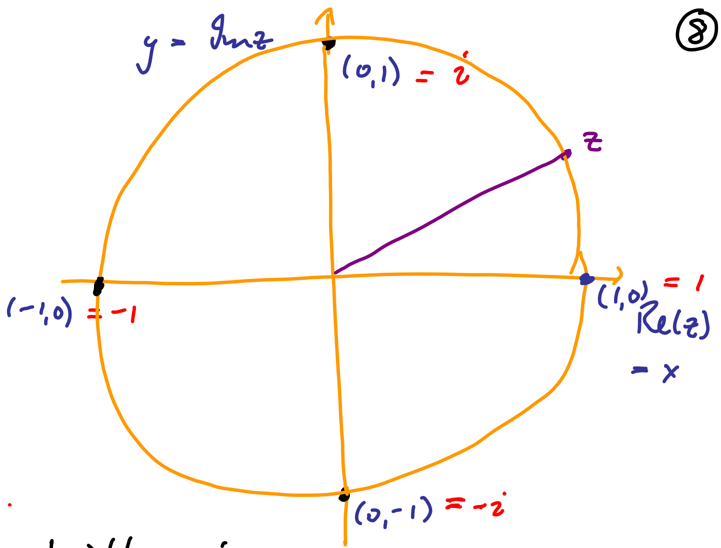
$$z = x + iy$$

$$z = \overset{1}{|z|}(\cos\varphi + i \sin\varphi)$$

$$= \cos\varphi + i\sin\varphi$$

$$= e^{i\varphi} \qquad (1)$$

werden wir jetzt zeigen.



Es gelten nämlich folgende Reihenentwicklungen:

$$e^w = \sum_{n=0}^{\infty} \frac{1}{(n!)} w^n = 1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \dots \qquad (2)$$

$n! \equiv 1 \cdot 2 \cdot 3 \cdots n$

$0! \equiv 1$

$$\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1} = w - \frac{1}{3!}w^3 + \frac{1}{5!}w^5 - \dots \qquad (3)$$

$$\cos w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} = 1 - \frac{1}{2!}w^2 + \frac{1}{4!}w^4 - \dots \qquad (4)$$

(Taylor-Reihen)

Konsistenzcheck: Wir wissen: $\frac{d}{dw} e^w = e^w$. (9)

Gilt das für (8.2)?

$$\frac{d}{dw} e^w \overset{(8.2)}{=} \frac{d}{dw} \sum_{n=0}^{\infty} \frac{1}{n!} w^n = \sum_{n=0}^{\infty} \underbrace{\frac{d}{dw} \frac{1}{n!} w^n}_{\frac{1}{\cancel{n}(n-1)!} \cancel{n} w^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} w^{n-1}$$

$$\overset{n-1=m}{=} \sum_{m=0}^{\infty} \frac{1}{m!} w^m \overset{(8.2)}{=} e^w$$

$$n! = n(n-1)!$$

$$n(n-1)(n-2)\cdots 1 = n(n-1)\cdot(n-2)(n-3)\cdots 1$$

Analog: wir wissen: $\frac{d}{dw} \sin w = \cos w$; gilt das für (8.3)? (8.4)?

$$\frac{d}{dw} \sin w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \underbrace{\frac{d}{dw} w^{2n+1}}_{(2n+1)w^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} = \cos w$$

Analog: $\frac{d}{dw} \cos w = -\sin w$

Def von $n!$ $\qquad n! = \begin{cases} 1\cdot 2\cdot 3 \cdots n & \forall\, n \neq 0,\ n \in \mathbb{N} \\ 1 & \forall\, n = 0 \end{cases}$ (10)

Betrachte nun (8.2) mit Argument iw:

$$e^{iw} = \sum_{n=0}^{\infty} \frac{1}{(n!)} (iw)^n$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n)!} (i)^{2n} w^{2n}}_{\text{geraden Potenzen v. } w} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (i)^{2n+1} w^{2n+1}}_{\text{ungeraden Potenzen}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} (-1)^n w^{2n} + i \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n w^{2n+1}$$

$$\overset{(8.4)}{=} \cos w + i \sin w = e^{iw}$$

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i \cdot i^2 = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

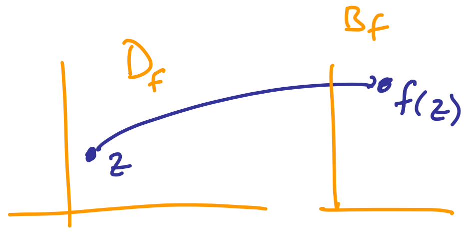
$i^5 = i \cdot i^4 = i$

$i^{2n} = (i^2)^n = (-1)^n$

$i^{2n+1} = i \cdot i^{2n} = i(-1)^n$

Wie zeichnet man eine komplexe Funktion?

Was ist überhaupt eine komplexe Funktion?



$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

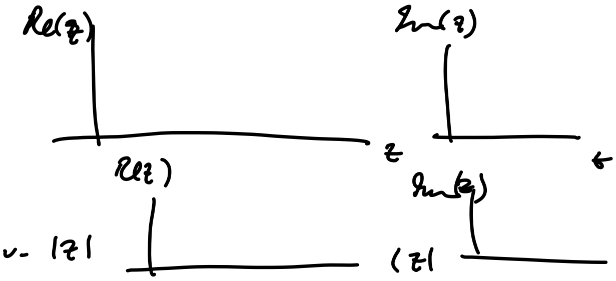
zB: $\quad z \longrightarrow f(z) = e^{z}$

Es ist nicht möglich, in einem "2D"-Plot alle Info. über $z \to f(z)$ zu skizzieren:

Übliche Darstellungen: $\forall$ z reell:

oder $\forall$ z Im.

oder: festes φ, als Fkt. v. $|z|$



Skizziere e^{iw} $\forall$ $w \in \mathbb{R}$:

$z = x + iy \qquad \operatorname{Im}(z) = y$

$\operatorname{Re}(e^{iw}) = \cos w \qquad \operatorname{Im}(e^{iw}) = \sin w$

$$e^{iw} : \mathbb{R} \to \mathbb{C}$$
$$w \to e^{iw}$$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$e^{x+iy} : \mathbb{C} \longrightarrow \mathbb{C}$$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$e^{i(x+iy)} = \cos(x+iy) + i \sin(x+iy)$$

$$= \cos x \cos(iy) - \sin x \sin(iy) \quad (1)$$

$$+ i\left[\sin x \cos iy + \cos x \sin iy\right] \quad (2)$$

$$= \underline{\cos x} \overset{a}{\cosh y} - \sin x \, i \overset{b}{\sinh y} \quad (3)$$

$$+ i\left(\sin x \overset{c}{\cosh y} + \underline{\cos x} \, i \overset{d}{\sinh y}\right) \quad (4)$$

$$= \cos x\left(\overset{a}{\cosh y} - \overset{d}{\sinh y}\right) \quad (5)$$

$$+ i \sin x \left(\overset{c}{\cosh y} - \overset{b}{\sinh y}\right) \qquad e^{i(x+iy)} \quad (6)$$

$$= (\cos x + i \sin x)(\cosh y - \sinh y) = e^{ix} \cdot e^{-y} \quad (7)$$

(4)

$$e^{iw} = \cos w + i \sin w \quad (1)$$

$$e^{-iw} = \cos(-w) + i \sin(-w) \quad (2)$$

$$e^{-iw} = \cos w - i \sin w \quad (3)$$

$$\frac{(1)+(3)}{2}: \quad \frac{e^{iw} + e^{-iw}}{2} = \cos w \quad (4) \qquad \frac{1}{i} = -i$$

$$1 = i \cdot \frac{1}{i} = i \cdot (-i) = 1$$

$$\frac{(1)-(3)}{2i}: \quad \frac{e^{iw} - e^{-iw}}{2i} = \sin w \quad (5)$$

$$(4): w = iv: \quad \cos(iv) = \left(e^{i(iv)} + e^{-i(iv)}\right)/2 = \frac{e^{-v} + e^{v}}{2} = \cosh v$$

$$(5): w = iv: \quad \sin(iv) = \frac{e^{-v} - e^{+v}}{2i} = i \tfrac{1}{2}\left(e^{v} - e^{-v}\right) = i \sinh v$$

$$\sin(iv) = \frac{e^{i(iv)} - e^{-i(iv)}}{i2}$$

$i^2 = -1$

$-i \cdot i = +1$

$$= \frac{e^{-v} - e^{v}}{i2} = \frac{1}{i} \, \frac{1}{2}(e^{-v} - e^{v})$$

$$= -i \, \frac{1}{2}(e^{-v} - e^{v})$$

$$= i \, \underbrace{\frac{1}{2}(e^{v} - e^{-v})}_{\equiv}$$

def.

$$= i \sinh(v)$$

$$e^{x} = \cosh x + \sinh x = \frac{1}{2}(e^{x} + e^{-x}) + \frac{1}{2}(e^{x} - e^{-x})$$

$$e^{-x} = \cosh x - \sinh x$$

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Euler-Formel für $w = \pi$:

$$e^{iw} = \cos w + i \sin w$$

$$e^{i\pi} = \cos\pi + i \sin\pi$$

$$= -1 + i \cdot 0$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$i y_1 = y_1 i$

$z = x + iy$

$z_1 \cdot z_2$

$(x_1 + iy_1)(x_2 + iy_2)$

$$= x_1 \cdot x_2 + \underbrace{(iy_1)(iy_2)}_{i^2 y_1 y_2} + i y_1 x_2 + x_1 i y_2$$