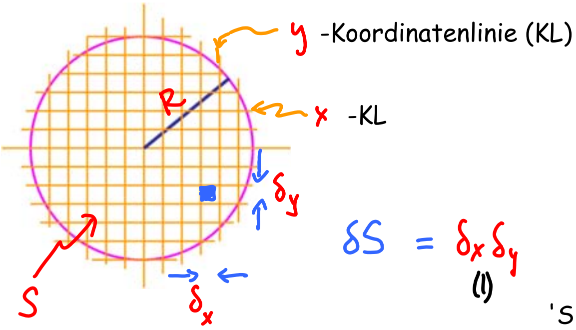
C4.3 Integration in krummlinigen Koordinaten

Falls ein System Symmetrien hat (z.B. invariant unter Rotationen um eine Symmetrie-Achse), lassen sich Integrale durch Nutzung krummliniger Koordinaten einfacher berechnen.

Beispiel: Kreisfläche

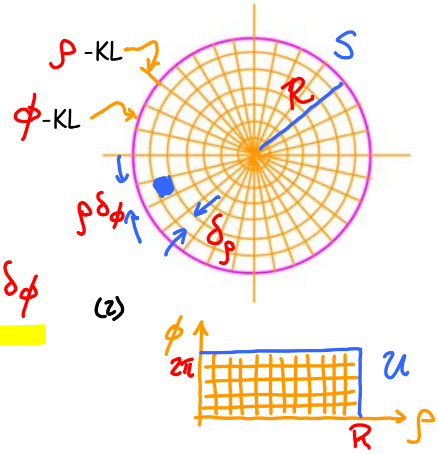


$$\delta S = \delta_x \delta_y \quad (1)$$

Flächen-element

$$\delta S = \rho\, \delta_\rho \delta_\phi \quad (2)$$

'S' für 'surface'



Nach geeigneter Koordinatentransformation wird aus der Scheibe

$$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$

ein 'Rechteck': $U = \left\{(\rho,\phi) \in \mathbb{R}^2 : \begin{array}{l} 0 < \rho \le R; \\ 0 \le \phi < 2\pi \end{array}\right\}$ (3)

Kreis-fläche:

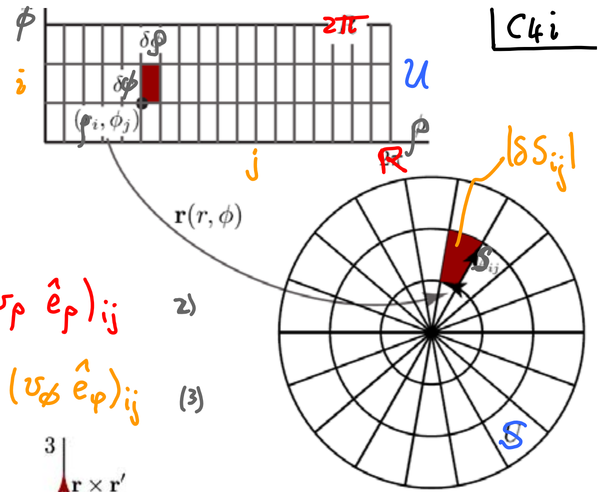
$$A = \iint_S \overbrace{dx\,dy}^{(1)} \cdot 1 \overset{(2)}{=} \iint_U \overbrace{\rho\, d\phi\, d\rho}^{(2)} = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi = \frac{1}{2} R^2 \cdot 2\pi = \pi R^2 \quad (4)$$

(Kartesisches Flächenelement; Polares Flächenelement)

[eleganter als auf Seite C4e !]

2D Integral in Polarkoordinaten

$$\int_S dS\, f(\vec{r}) \simeq \sum_{ij} |\delta S_{ij}|\, f(\rho_i, \phi_j) \quad (1)$$

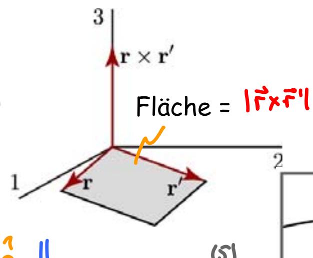


Fläche $|\delta S_{ij}|$ wird aufgespannt durch:

$$\vec{r}(\rho_i + \delta_\rho, \phi_j) - \vec{r}(\rho_i, \phi_j) = \delta_\rho \partial_\rho \vec{r}(\rho_i, \phi_i) \overset{(V5f.2)}{=} \delta_\rho (v_\rho\, \hat{e}_\rho)_{ij} \quad (2)$$

$$\vec{r}(\rho_i, \phi_j + \delta_\phi) - \vec{r}(\rho_i, \phi_j) = \delta_\phi \partial_\phi \vec{r}(\rho_i, \phi_i) \overset{(V5f.4)}{=} \delta_\phi (v_\phi\, \hat{e}_\varphi)_{ij} \quad (3)$$

$$|\delta S_{ij}| = \delta_\rho \delta_\phi \left\| (v_\rho\, \hat{e}_\rho)_{ij} \times (v_\phi\, \hat{e}_\phi)_{ij} \right\| \quad (4)$$

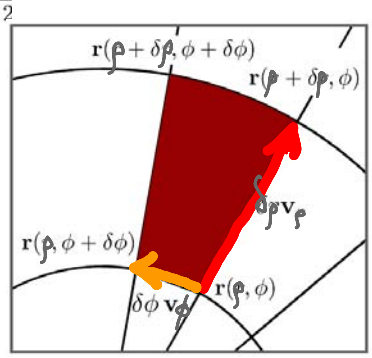


$$\int_D dS\, f(\vec{r}) = \int_0^R d\rho \int_0^{2\pi} d\phi\, f(\rho, \phi)\; |\underset{\overset{\shortparallel}{1}}{v_\rho}\, \underset{\overset{\shortparallel}{\rho}}{v_\phi}|\; \underbrace{\|\hat{e}_\rho \times \hat{e}_\phi\|}_{=1} \quad (5)$$

(V5I.8,9): $v_\rho = 1$, $v_\phi = \rho$ (6) (V5I.12)

$$= \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, f(\rho, \phi) \quad (7)$$

Flächenelement in Polarkoordinaten: $'dS = \rho\, d\rho\, d\phi'$ (8)

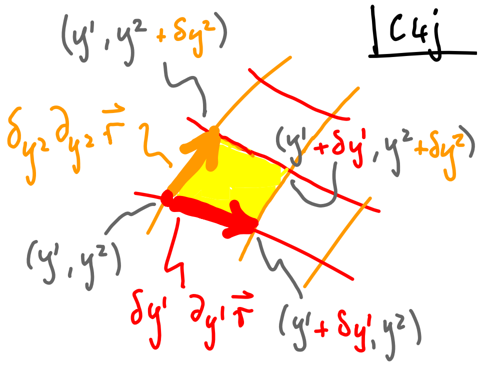


C4j

Allgemeine Koordinatentransformation in 2D

$$\vec r: \ \mathcal U \subset \mathbb R^2 \longrightarrow S \subset \mathbb R^2$$

$$\vec y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \longmapsto \vec r(\vec y) = \begin{pmatrix} x^1(y^1, y^2) \\ x^2(y^1, y^2) \end{pmatrix} \qquad (1)$$

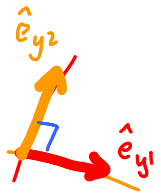


Flächenelement wird aufgespannt durch: $\delta_{y^1}\partial_{y^1}\vec r$, $\delta_{y^2}\partial_{y^2}\vec r$ (2)

$$\delta S \overset{(L4b.1)}{=} \delta_{y^1}\,\delta_{y^2}\,\|\partial_{y^1}\vec r \times \partial_{y^2}\vec r\| \qquad (3)$$

2D-Integral: $$\int_S dS\, f(\vec r) = \int_{\mathcal U} dy^1 \int dy^2\, \|\partial_{y^1}\vec r \times \partial_{y^2}\vec r\|\, f(\vec r(y^1, y^2)) \qquad (4)$$

(Vsg.4.5) $$\partial_{y^1}\vec r \times \partial_{y^2}\vec r = v_{y^1}\hat e_{y^1} \times v_{y^2}\hat e_{y^2} \qquad (5)$$



Für krummlinig-orthogonale Koordinaten gilt: $$\|\hat e_{y^1} \times \hat e_{y^2}\| = 1 \qquad (6)$$

Flächenelement: $$dS = dy^1\, dy^2\, v_{y^1}\, v_{y^2} \qquad (7)$$

Bsp: Polarkoord.

$y^1 = \rho$, $y^2 = \phi$

$v_{y^1} = 1$, $v_{y^2} = \rho$

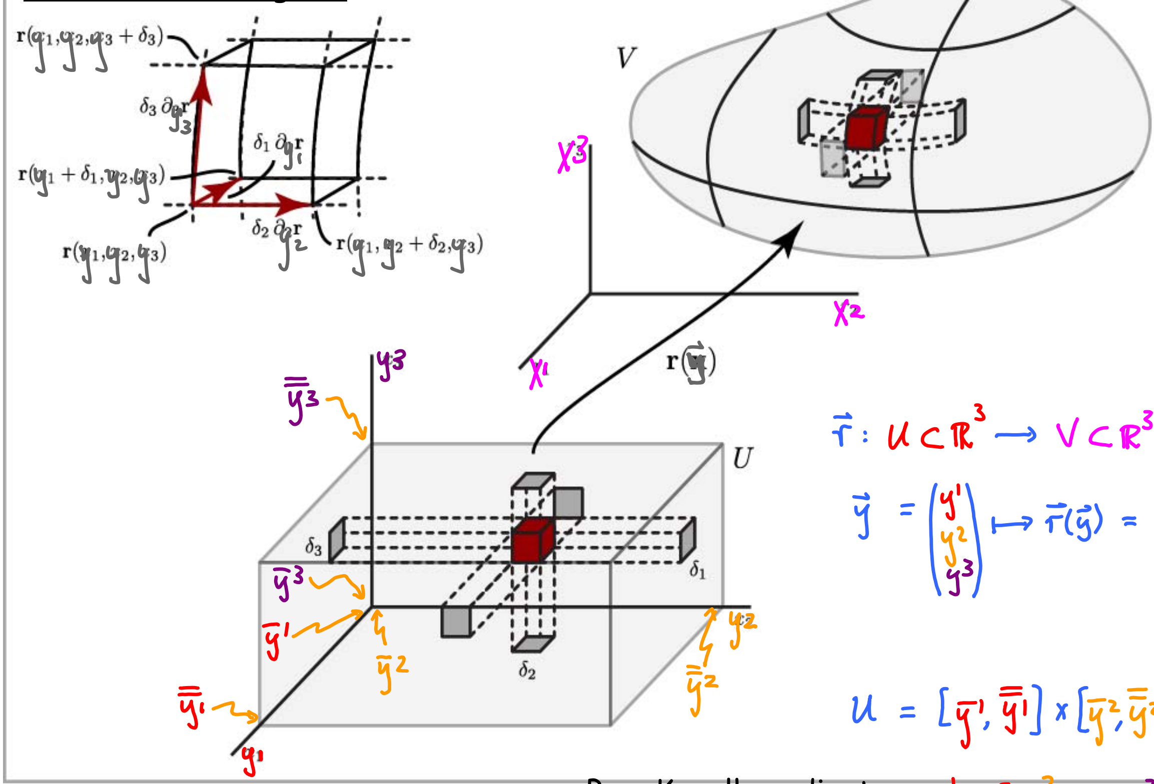
$\hat e_{y^1} = \hat e_\rho$, $\hat e_{y^2} = \hat e_\rho$

$\hat e_{y^1} \times \hat e_{y^2} = 1$

$dS \overset{(7)}{=} d\rho\, d\phi \cdot \rho$

C4k

C4.4 Volumenintegrale



$$\vec r: \ \mathcal U \subset \mathbb R^3 \longrightarrow V \subset \mathbb R^3$$

$$\vec y = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \longmapsto \vec r(\vec y) = \begin{pmatrix} x^1(y^1, y^2, y^3) \\ x^2(y^1, y^2, y^3) \\ x^3(y^1, y^2, y^3) \end{pmatrix}$$

$$\mathcal U = [\bar y^1, \bar{\bar y}^1] \times [\bar y^2, \bar{\bar y}^2] \times [\bar y^3, \bar{\bar y}^3]$$

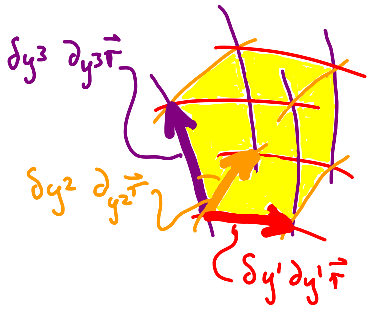
Bsp: Kugelkoordinaten: $y^1 = r$, $y^2 = \theta$, $y^3 = \phi$

Ball mit Radius R: $\mathcal U =]0, R] \times [0, \pi] \times [0, 2\pi[$

Allgemeine Koordinatentransformation in 3D

C4l

$$\vec{r}: \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3$$
$$\vec{y} \longmapsto \vec{r}(y^1, y^2, y^3) \qquad (1)$$



Volumenelement:

Volumen des Parallelelipeds = Spatprodukt (L4m.4):

$$\delta V = \left| \left(\frac{\partial \vec{r}}{\partial y^1} \times \frac{\partial \vec{r}}{\partial y^2} \right) \cdot \frac{\partial \vec{r}}{\partial y^3} \right| \delta y^1 \, \delta y^2 \, \delta y^3 \qquad (2)$$

3D-Integral:

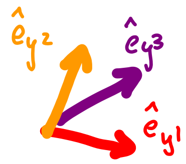
$$\int_V dV f(\vec{r}) = \int_{\underline{y}^1}^{\bar{y}^1} dy^1 \int_{\underline{y}^2}^{\bar{y}^2} dy^2 \int_{\underline{y}^3}^{\bar{y}^3} dy^3 \, \left\| \left(\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r} \right) \cdot \partial_{y^3}\vec{r} \right\| \, f(\vec{r}(y^1, y^2, y^3)) \qquad (3)$$

(V5g.4,5):

$$\left(\partial_{y^1}\vec{r} \times \partial_{y^2}\vec{r} \right) \cdot \partial_{y^3}\vec{r} = \left(v_{y^1} \hat{e}_{y^1} \times v_{y^2} \hat{e}_{y^2} \right) \cdot \left(v_{y^3} \hat{e}_{y^3} \right) \qquad (4)$$

Für krummlinig-orthogonale Koordinaten gilt:

$$\left| \left(\hat{e}_{y^1} \times \hat{e}_{y^2} \right) \cdot \hat{e}_{y^3} \right| = 1 \qquad (5)$$



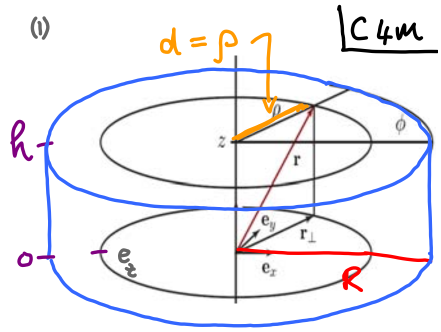
Volumenelement:

$$dV = dy^1 \, dy^2 \, dy^3 \, v_{y^1} \, v_{y^2} \, v_{y^3} \qquad (6)$$

Beispiel 1: Zylinderkoordinaten

$$y^1 = \rho, \quad y^2 = \phi, \quad y^3 = z \qquad (1)$$

C4m



(V5l.1):
$$\vec{r} = \rho \cos\phi \, \hat{e}_1 + \rho \sin\phi \, \hat{e}_2 + z \, \hat{e}_3 \qquad (2)$$

(V5l.8-10):
$$v_\rho = 1, \quad v_\varphi = \rho, \quad v_z = 1 \qquad (3)$$

(C4l.6):
$$dV = d\rho \, d\phi \, dz \cdot \rho \qquad (4)$$

Volumen eines Zylinders:
$$V = \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz = \tfrac{1}{2} R^2 \cdot 2\pi \cdot h = \pi R^2 h \qquad (5)$$

Trägheitsmoment eines homogenen Zylinders:

Dichte = ρ_0 = Masse/Volumen = $\dfrac{M}{\pi R^2 h}$ (6)

(nicht mit Radius zu verwechseln!)

Abstand v. Symmetrieachse: $d_\perp(\vec{r}) = \rho$ (7)

$$I \overset{\text{(C4f.3)}}{=} \int_V dV \, \underbrace{\rho(\vec{r})}_{(6):\ \rho_0} \, \underbrace{d_\perp^2(\vec{r})}_{(7):\ \rho^2} \overset{\text{(C4l.6)}}{=} \int_0^R d\rho \cdot \rho \int_0^{2\pi} d\phi \int_0^h dz \; \rho_0 \cdot \rho^2 = \underbrace{\rho_0}_{\overset{(6)}{=}\frac{M}{\pi R^2 h}} \underbrace{\int_0^R d\rho \, \rho^3}_{\frac{1}{4} R^4} \underbrace{\int_0^{2\pi} d\phi \cdot 1}_{2\pi} \underbrace{\int_0^h dz \cdot 1}_{h} \qquad (8)$$

$$= \tfrac{1}{2} M R^2 \qquad (9)$$

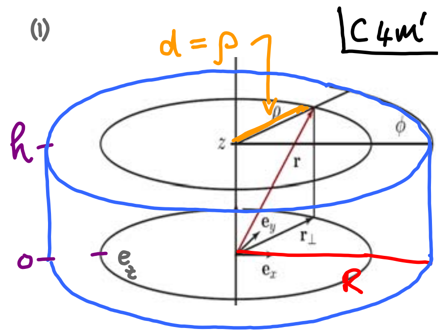
Beispiel 2: Volumen eines Kegels $y^1 = \rho$, $y^2 = \phi$, $y^3 = z$ (1)

C4m'

(V5l.1): $\vec{r} = \rho\cos\phi\,\hat{e}_1 + \rho\sin\phi\,\hat{e}_2 + z\,\hat{e}_3$ (2)

(V5l.8-10): $v_\rho = 1$, $v_\varphi = \rho$, $v_z = 1$ (3)

(C4l.6): $dV = d\rho\, d\phi\, dz \cdot \rho$ (4)

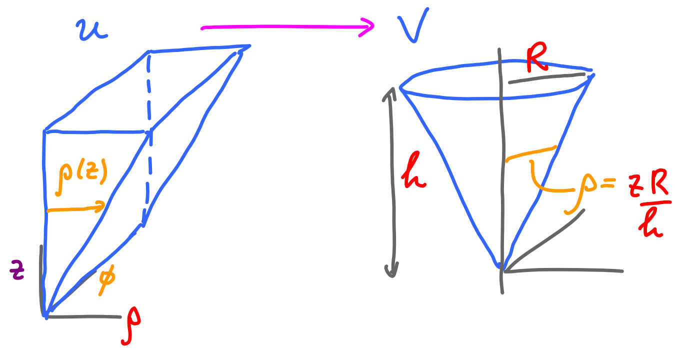


$$V = \int_0^h dz \int_0^{zR/h} d\rho \cdot \rho \int_0^{2\pi} d\phi$$

$$= \int_0^h dz \; \tfrac{1}{2}\rho^2 \Big|_0^{zR/h} \; 2\pi$$

$$= \pi \left(\frac{R}{h}\right)^2 \underbrace{\int_0^h dz \; z^2}_{\frac{1}{3}h^3}$$

$$= \frac{\pi}{3} h R^2$$



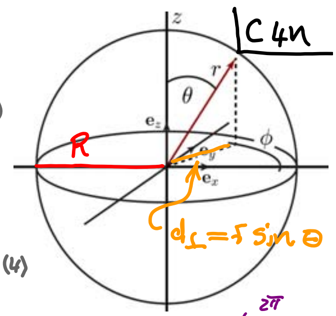
Beispiel 3: Kugelkoordinaten $y^1 = r$, $y^2 = \theta$, $y^3 = \phi$ (1)

C4n

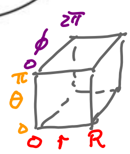
(V5m.1): $\vec{r} = r\sin\theta\cos\phi\,\hat{e}_1 + r\sin\theta\sin\phi\,\hat{e}_2 + r\cos\theta\,\hat{e}_3$ (2)

(V5m.8-10): $v_r = 1$, $v_\theta = r$, $v_\phi = r\sin\theta$ (3)

(C4l.6): $dV = dr\, d\theta\, d\phi \cdot 1 \cdot r \cdot r\sin\theta = dr\, d\theta\, d\phi\, r^2 \sin\theta$ (4)



Volumen einer Kugel: $V = \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = \frac{1}{3} R^3 \underbrace{(-\cos\theta)\Big|_0^\pi}_{2} 2\pi = \frac{4}{3}\pi R^3$ (5)



Trägheitsmoment einer homogenen Kugel:

Dichte = ρ_0 =Masse/Volumen = $M/(\frac{4}{3}\pi R^3)$ (6) Abstand v. Symmetrieachse: $d(\vec{r}) = r\sin\theta$ (7)

$$I \overset{(C4f.3)}{=} \int_V dV \overbrace{\rho(\vec{r})}^{\rho_0} \overbrace{d_\perp^2(\vec{r})}^{(r\sin\theta)^2} = \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \cdot \rho_0\, r^2 (\sin\theta)^2 \quad (8)$$

$$= \underbrace{\rho_0}_{\overset{(6)}{} \frac{M}{4\pi/3\, R^3}} \underbrace{\int_0^R dr\, r^4}_{\frac{1}{5}R^5} \underbrace{\int_0^\pi d\theta\, (\sin\theta)^3}_{\cos\theta = u} \underbrace{\int_0^{2\pi} d\phi \; 1}_{2\pi} = \frac{2}{5} M R^2 \quad (9)$$

$\cos\theta = u$: $\int_{-1}^{1} du\,(1-u^2) = \left[u - \frac{1}{3}u^3\right]_{-1}^{1} = 2\left[1 - \frac{1}{3}\right] = \frac{4}{3}$

$$\boxed{\begin{aligned} \cos\theta &= u \\ -\sin\theta &= \frac{du}{d\theta} \\ \int_0^\pi d\theta \sin\theta &= +\int_{-1}^{+1} du \end{aligned}}$$

Integrationsmaß = Jakobi-Determinante (zur Kenntnisnahme)

2D-Integral: $$\int_S dS\, f(\vec r) \overset{(j.4)}{=} \int dy^1 \int dy^2 \, \| \partial_{y^1}\vec r \times \partial_{y^2}\vec r \| \; f(\vec r(y^1, y^2)) \qquad (1)$$

$$\partial_{y^1}\vec r \times \partial_{y^2}\vec r = \begin{pmatrix} \partial_{y^1} x^1 \\ \partial_{y^1} x^2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \partial_{y^2} x^1 \\ \partial_{y^2} x^2 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \partial_{y^1} x^1 \, \partial_{y^2} x^2 \\ - \partial_{y^1} x^2 \, \partial_{y^2} x^1 \end{bmatrix} \qquad (2)$$

$$\Rightarrow \quad dS = dy^1\, dy^2 \, \underbrace{\left| \frac{\partial x^1}{\partial y^1}\frac{\partial x^2}{\partial y^2} - \frac{\partial x^2}{\partial y^1}\frac{\partial x^1}{\partial y^2} \right|}_{\equiv \left| \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} \right|} \qquad (3)$$

'Jakobi-Determinante', 'Funktional-Determinante'

3D-Integral: $$\int_V dV\, f(\vec r) \overset{(l.3)}{=} \int dy^1 \int dy^2 \int dy^3 \, \| (\partial_{y^1}\vec r \times \partial_{y^2}\vec r) \cdot \partial_{y^3}\vec r \| \; f(\vec r(y^1, y^2, y^3)) \qquad (4)$$

Spatprodukt lässt sich durch eine 'Determinante' (siehe L6.1) ausdrucken:

$$\left(\frac{\partial \vec r}{\partial y^1} \times \frac{\partial \vec r}{\partial y^2}\right) \cdot \frac{\partial \vec r}{\partial y^3} \overset{(C4P.4)}{=} \det\left[\frac{\partial \vec r}{\partial y^1}, \frac{\partial \vec r}{\partial y^2}, \frac{\partial \vec r}{\partial y^3}\right] \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \qquad (5)$$

'Jakobi-Determinante', 'Funktional-Determinante'

Kurznotation

Also: Volumenelement: $$dV \overset{(3)}{=} dy^1\, dy^2\, dy^3 \left| \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \right| \qquad (6)$$

Spatprodukt durch Determinante ausgedrückt (zur Kenntnisnahme)

(zum später Lesen, nach Abschnitt L6.1)

$$[\vec a \times \vec b] \cdot \vec c = \left[\begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \times \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}\right] \cdot \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \begin{pmatrix} a^2 b^3 - a^3 b^2 \\ a^3 b^1 - a^1 b^3 \\ a^1 b^2 - a^2 b^1 \end{pmatrix} \cdot \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix} = \begin{matrix} (a^2 b^3 - a^3 b^2) c^1 \\ + (a^3 b^1 - a^1 b^3) c^2 \\ + (a^1 b^2 - a^2 b^1) c^3 \end{matrix} \qquad (1)$$

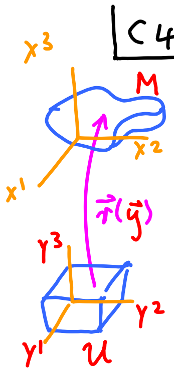
$$\det|\vec a, \vec b, \vec c| \equiv \begin{vmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} c^1 & a^1 & b^1 \\ c^2 & a^2 & b^2 \\ c^3 & a^3 & b^3 \end{vmatrix} \equiv \begin{matrix} (a^2 b^3 - a^3 b^2) c^1 \\ + (a^3 b^1 - a^1 b^3) c^2 \\ + (a^1 b^2 - a^2 b^1) c^3 \end{matrix} \qquad (2)$$

$$(1) = (2) \Rightarrow \qquad [\vec a \times \vec b] \cdot \vec c = \det|\vec a, \vec b, \vec c| \qquad (3)$$

Funktional-Determinante: $$\left\| \left(\frac{\partial \vec r}{\partial y^1} \times \frac{\partial \vec r}{\partial y^2}\right) \cdot \frac{\partial \vec r}{\partial y^3} \right\| = \begin{vmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \frac{\partial x^1}{\partial y^3} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \frac{\partial x^2}{\partial y^3} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} & \frac{\partial x^3}{\partial y^3} \end{vmatrix} \equiv \left| \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \right| \qquad (4)$$

C4.5 Krummlinige Integration in n Dimensionen (nur zur Kenntnisnahme)

C4q



$$\vec r : \; \mathcal{U} \subset \mathbb{R}^n \longrightarrow M \subset \mathbb{R}^n$$

$$\vec y \;\longmapsto\; \vec r(y^1, \dots, y^n) = \begin{bmatrix} x^1(y^1, \dots, y^n) \\ \vdots \\ x^n(y^1, \dots, y^n) \end{bmatrix} \qquad (1)$$

n-dimensionales 'Volumenelement':

$$dV^n = dy^1 \dots dy^n \cdot \left| \frac{\partial(x^1, \dots, x^n)}{\partial(y^1, \dots, y^n)} \right| \qquad (2)$$

'Jakobi-Determinante', 'Funktionaldeterminante':

$$\frac{\partial(x^1, \dots, x^n)}{\partial(y^1, \dots, y^n)} \equiv \det \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \cdots & \frac{\partial x^1}{\partial y^n} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \cdots & \frac{\partial x^2}{\partial y^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x^n}{\partial y^1} & \frac{\partial x^n}{\partial y^2} & \cdots & \frac{\partial x^n}{\partial y^n} \end{bmatrix} \qquad (3)$$

'Determinante'

n-dimensionales Integral:

$$\int_M dx_1 \dots \int dx_n \, f(x_1, \dots, x_n) = \int_{\mathcal{U}} dy^1 \dots \int dy^n \left| \frac{\partial(x^1, \dots, x^n)}{\partial(y^1, \dots, y^n)} \right| f(x^1(\vec y), \dots, x^n(\vec y)) \qquad (4)$$

Das ist Verallgemeinerung der Substitutionsregel, (C2g.5): $\int dx\, f(x) = \int dy \, \frac{dx}{dy} f(x(y))$ (5)

Zusammenfassung: Allgemeine Koordinatentransformation in 2D

ZC4b

$$\vec r : \; \mathcal{U} \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^2, \quad \vec y \longmapsto \vec r(\vec y) = \begin{pmatrix} x^1(\vec y) \\ x^2(\vec y) \end{pmatrix} \qquad (1)$$

$$\iint_S dx^1 \, dx^2 \, f(\vec r) = \iint_{\mathcal{U}} dy^1 \, dy^2 \left| \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} \right| f(\vec r(y^1, y^2)) \qquad (2)$$

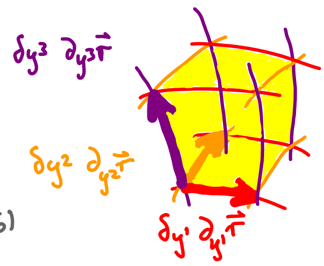
Jakobi-Determinante

$$\left| \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} \right| = \| \partial_{y^1} \vec r \times \partial_{y^2} \vec r \| = v_{y^1} v_{y^2} \qquad (3)$$

für krummlinig-orthogonale Koord.

Polar: $dS = \rho \, d\rho \, d\phi$ (4)

Zusammenfassung: Allgemeine Koordinatentransformation in 3D



$$\vec r : \; \mathcal{U} \subset \mathbb{R}^3 \longrightarrow V \subset \mathbb{R}^3, \quad \vec y \longmapsto \begin{pmatrix} x^1(\vec y) \\ x^2(\vec y) \\ x^3(\vec y) \end{pmatrix} \qquad (5)$$

$$\iiint_V dx^1 \, dx^2 \, dx^3 \, f(\vec r) = \iiint_{\mathcal{U}} dy^1 \, dy^2 \, dy^3 \left| \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \right| f(\vec r(y^1, y^2, y^3)) \qquad (6)$$

Jakobi-Determinante

$$\left| \frac{\partial(x^1, x^2, x^3)}{\partial(y^1, y^2, y^3)} \right| = \| (\partial_{y^1} \vec r \times \partial_{y^2} \vec r) \cdot \partial_{y^3} \vec r \| = v_{y^1} v_{y^2} v_{y^3} \qquad (7)$$

für krummlinig-orthogonale Koord.

Zylinder: $dV = \rho \, d\rho \, d\phi \, dz$ (8)

Kugel: $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$ (9)