Fakultät für Physik
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http://homepages.physik.uni-muenchen.de/~vondelft/Lehre/15r/

Sheet 02.4: Vector Spaces, Euclidean Spaces

Posted: Friday, 23.10.15 Due: Friday, 30.10.2015, 13:00 Central Tutorial: 04.11.15
[2](E/M/A) means: problem counts 2 points and is easy/medium hard/advanced

Example Problem 1: $\sqrt{1-x^2}$ Integrals by trigonometric substitution [4]
Points: (a)[1](M); (b)[3](A)

The sine and cosine functions satisfy the following identities:

$$\frac{\mathrm{d}}{\mathrm{d}y}\sin(y) = \cos(y), \qquad \frac{\mathrm{d}}{\mathrm{d}y}\cos(y) = -\sin(y), \qquad \cos^2(y) = 1 - \sin^2(y).$$

The last of these is useful for solving integrals that contain $\sqrt{1-x^2}$ by using the trigonometric substitution $x = \sin(y)$, since $\sqrt{1-x^2} = \cos(y)$.
Calculate the following integrals for $|z| < 1$; check your results by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.

(a) $I(z) = \int_0^z \mathrm{d}x \, \frac{1}{\sqrt{1-x^2}}$. $\left[\text{Check your result: } I\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}. \right]$

(b) $I(z) = \int_0^z \mathrm{d}x \, \sqrt{1-x^2}$. $\left[\text{Check your result: } I\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{8} + \frac{1}{4}. \right]$

Hint: The $\cos^2 y$ integral emerging after substitution can be solved by integrating by parts!

Example Problem 2: Linear independence [3]
Points: (a)[2](M); (b)[1](M)

(a) Are the vectors $\mathbf{v}_1 = (0,1,2)^T$, $\mathbf{v}_2 = (1,-1,1)^T$ and $\mathbf{v}_3 = (2,-1,4)^T$ linearly independent?

(b) If yes (or if no), find a vector $\mathbf{v}_2'$ such that $\mathbf{v}_1$, $\mathbf{v}_2'$ and $\mathbf{v}_3$ are not (or are) linearly independent, and show explicitly that they have this property.

Example Problem 3: Projection onto an orthonormal basis [2]
Points: (a)[1](E); (b)[1](E)

(a) Show that the vectors $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(1,1)^T$, $\mathbf{e}_2' = \frac{1}{2}(1,-1)^T$ form an orthonormal basis for $\mathbb{R}^2$.

(b) Express the vector $\mathbf{w} = (-2,3)^T$ in the form $\mathbf{w} = \mathbf{e}_1' w^1 + \mathbf{e}_2' w^2$, by computing its components w^i with respect to the basis $\{\mathbf{e}_i'\}$ through projection onto the basis vectors.

Example Problem 4: Gram-Schmidt procedure [2]
Points: [2](E)

Use the Gram-Schmidt procedure for the following set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to construct an orthonormal set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ with the same span and with $\mathbf{e}'_1 || \mathbf{v}_1$.

$$\mathbf{v}_1 = (1, -2, 1)^T \,, \qquad \mathbf{v}_2 = (1, 1, 1)^T \,, \qquad \mathbf{v}_3 = (0, 1, 2)^T \,.$$

Example Problem 5: Non-orthogonal basis and metric [4]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[1](M)

Consider the vectors $\mathbf{v}_1 = \binom{2}{0}$ and $\mathbf{v}_2 = \binom{1}{1}$, written as column vectors in the standard basis of $\mathbb{R}^2$.

(a) Write the standard basis vector $\mathbf{e}_1 = \binom{1}{0}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Ditto for $\mathbf{e}_2 = \binom{0}{1}$. Do $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$?

(b) Let $\mathbf{x} = \mathbf{v}_1 x^1 + \mathbf{v}_2 x^2$ and $\mathbf{y} = \mathbf{v}_1 y^1 + \mathbf{v}_2 y^2$ be two vectors in $\mathbb{R}^2$, whose components w.r.t. $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by $x^1 = 3$, $x^2 = -4$ and $y^1 = -1$, $y^2 = 3$ respectively. Express $\mathbf{x}$ and $\mathbf{y}$ as column vectors in the standard basis of $\mathbb{R}^2$ and compute their scalar product, $\mathbf{x} \cdot \mathbf{y}$.

(c) Find the components of the metric $g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ explicitly (specifically: find g_{11}, g_{12}, g_{21} and g_{22}).

(d) Now calculate the scalar product of $\mathbf{x}$ and $\mathbf{y}$ using the formula $\mathbf{x} \cdot \mathbf{y} = x^i g_{ij} y^j = x_j y^j$, with $x_j = x^i g_{ij}$, and carry out the sum over i and j explicitly. [Check: is the result consistent with that from (b)?]

Example Problem 6: Vector space of monomials of degree 2 [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

The vector space of all real functions is infinite-dimensional. However, if only functions of a prescribed form are considered, the corresponding vector space can be finite-dimensional. As an example, it is shown in this problem that the set of all monomials of degree 2 form a vector space of dimension 1, isomorphic to $\mathbb{R}$.
[Remark on the notation: In the context of the present problem on polynomials, x^2 means "x to the power of 2", in contrast to the notation that we have adopted elsewhere when discussing vectors, where x^k stands for the k component of the vector $\mathbf{x} = \sum_k \mathbf{v}_k x^k$ with respect to a basis of vectors $\{\mathbf{v}_k\}$. Every notational convention has exceptions!]

Let p_a denote a polynomial in the variable $x \in \mathbb{R}$ of degree 2, uniquely specified by the coefficient $x \in \mathbb{R}$:

$$p_a : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_a(x) \equiv a x^2 \,.$$

Let $P_2 = \{p_a | a \in \mathbb{R}^{n+1}\}$ denote the set of all such polynomials of degree 2. The natural definitions for adding such polynomials, or multiplying them by a scalar $c \in \mathbb{R}$, are

$$p_a \boldsymbol{+} p_b : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_a(x) + p_b(x) \,, \tag{1}$$

$$c \boldsymbol{\cdot} p_a : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto c\, p_a(x), \tag{2}$$

where on the right side the usual addition and multiplication in $\mathbb{R}$ is used.

(a) Show that the above addition and scalar multiplication imply the following composition rules in P_2,

Addition of polynomials:	$\boldsymbol{+}$:	$P_2 \times P_2 \to P_2,$	$(p_a, p_b) \mapsto p_a \boldsymbol{+} p_b \equiv p_{a+b}\,,$
Multiplication by a scalar:	$\cdot$:	$\mathbb{R} \times P_2 \to P_2,$	$(c, p_x) \mapsto c \cdot p_a \equiv p_{ca}\,,$

where $a + b$ and ca denote the usual addition and scalar multiplication in $\mathbb{R}$.

(b) Show that $(P_2, \boldsymbol{+}, \cdot)$ is an $\mathbb{R}$ vector space, and that it is isomorphic to $\mathbb{R}$.

(c) Specify a basis for $(P_2, \boldsymbol{+}, \cdot)$.

Example Problem 7: Vektor space of real functions [2]
Points: [2](M)

Let $F \equiv \{f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x)\}$ be the set of real functions. Show that $(F, \boldsymbol{+}, \cdot)$ is an $\mathbb{R}$ vector space, where the addition of functions, and their multiplication by a scalar, are defined as follows:

$$\boldsymbol{+} : F \times F \to F \qquad (f, g) \mapsto f \boldsymbol{+} g, \qquad \text{with} \qquad f \boldsymbol{+} g : x \mapsto \big[f \boldsymbol{+} g\big](x) \equiv f(x) + g(x) \tag{3}$$
$$\cdot : \mathbb{R} \times F \to F \qquad (\lambda, f) \mapsto \lambda \cdot f, \qquad \text{with} \qquad \lambda \cdot f : x \mapsto \big[\lambda \cdot f\big](x) \equiv \lambda f(x) \tag{4}$$

Remark regarding notation: It is important to distinguish the 'name' of a function, f, from the 'function value', $f(x)$, which it returns when evalutated at the argument x. The sum of the functions f and g is a function named $f \boldsymbol{+} g$. Eq. (3) states that its function value at x, denoted by $\big[f \boldsymbol{+} g\big](x)$ (square brackets indicate the function name), is by definition equal to $f(x) + g(x)$, the sum of the function values of f and g at x. The product of the number c and the function f yields a function named $c \cdot f$. Eq. (4) states that its function value at x, denoted by $\big[c \cdot f\big](x)$, is by definition equal to $cf(x)$, the product of c with the function value of f at x.

Example Problem 8: Real vector space with unconventional composition rules [Bonus]
Points: [2](M,Bonus)

The axioms that define a vector space can be satisfied in many different ways. These may involve unconventional definitions of vector addition and scalar multiplication. The purpose of the present problem is to illustrate this point.
For any $a \in \mathbb{R}$, let $V_a \equiv \{\mathbf{v}_x\}$ be a set whose elements $\mathbf{v}_x$, labelled by real numbers $x \in \mathbb{R}$, satisfy the following composition rules:

Addition:	$\boldsymbol{+}$:	$V_a \times V_a \to V_a,$	$(\mathbf{v}_x, \mathbf{v}_y) \mapsto \mathbf{v}_x \boldsymbol{+} \mathbf{v}_y \equiv \mathbf{v}_{x+y+a}$
Multiplication by a scalar:	$\cdot$:	$\mathbb{R} \times V_a \to V_a,$	$(\lambda, \mathbf{v}_x) \mapsto \lambda \cdot \mathbf{v}_x \equiv \mathbf{v}_{\lambda x + a(\lambda - 1)}$

The a and x labels, being real numbers, satisfy the usual addition and scalar multiplication rules of $\mathbb{R}$; e.g. in V_2 we have: $\mathbf{v}_3 \boldsymbol{+} \mathbf{v}_4 = \mathbf{v}_{3+4+2} = \mathbf{v}_9$ and $3 \cdot \mathbf{v}_4 = \mathbf{v}_{3 \cdot 4 + 2(3-1)} = \mathbf{v}_{16}$. Show that the triple $(V_a, \boldsymbol{+}, \cdot)$ represents an $\mathbb{R}$-vector space, with $\mathbf{v}_{-a}$ and 1 being the neutral elements for addition and scalar multiplication, respectively, and $\mathbf{v}_{-x-2a}$ the additive inverse of $\mathbf{v}_x$.

[Total Points for Example Problems: 20]

Homework Problem 1: $\sqrt{1+x^2}$ Integrals by hyperbolic substitution [4]
Points: (a)[1](M); (b)[3](A)

The 'hyperbolic sine' and 'hyperbolic cosine' functions, defined by

$$\sinh(y) = \tfrac{1}{2}(\mathrm{e}^y - \mathrm{e}^{-y}), \qquad \cosh(y) = \tfrac{1}{2}(\mathrm{e}^y + \mathrm{e}^{-y}),$$

satisfy the following identities:

$$\tfrac{\mathrm{d}}{\mathrm{d}y}\sinh(y) = \cosh(y), \qquad \tfrac{\mathrm{d}}{\mathrm{d}y}\cosh(y) = \sinh(y), \qquad \cosh^2(y) = 1 + \sinh^2(y).$$

The last of these is useful for solving integrals that contain $\sqrt{1+x^2}$ by using the trigonometric substitution $x = \sin(y)$, since $\sqrt{1+x^2} = \cosh(y)$.
Calculate the following integrals for $|z| < 1$; check your results by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.

(a) $I(z) = \displaystyle\int_0^z \mathrm{d}x \, \frac{1}{\sqrt{1+x^2}}$. [Check your result: $I\left(\frac{3}{4}\right) = \ln 2$.]

(b) $I(z) = \displaystyle\int_0^z \mathrm{d}x \, \sqrt{1+x^2}$. [Check your result: $I\left(\frac{3}{4}\right) = \ln\sqrt{2} + \frac{15}{32}$.]

Hint: The $\cosh^2 y$ integral emerging after substitution can be solved by integrating by parts!

Homework Problem 2: Linear independence [3]
Points: (a)[2](M); (b)[1](M)

(a) Are the vectors $\mathbf{v}_1 = (1,2,3)^T$, $\mathbf{v}_2 = (2,4,6)^T$ and $\mathbf{v}_3 = (-1,-1,0)^T$ linearly independent?

(b) If yes (or if no), find a vector $\mathbf{v}_2'$ such that $\mathbf{v}_1$, $\mathbf{v}_2'$ and $\mathbf{v}_3$ are not (or are) linearly independent, and show explicitly that they have this property.

Homework Problem 3: Projection onto an orthonormal basis [2]
Points: (a)[1](E); (b)[1](E)

(a) Show that the vectors $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(1,0,1)^T$, $\mathbf{e}_2' = \frac{1}{2}(-1,\sqrt{2},1)^T$ and $\mathbf{e}_3' = \frac{1}{2}(-1,-\sqrt{2},1)^T$ form an orthonormal basis in $\mathbb{R}^3$.

(b) Let $\mathbf{w} = \mathbf{e}_i' w^i$ be the decomposition of $\mathbf{w} = (1,2,3)^T$ in this basis. Find the components w^i.

Homework Problem 4: Gram-Schmidt procedure [2]
Points: [2](E)

Use the Gram-Schmidt procedure for the following set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to construct an orthonormal set $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ with the same span and with $\mathbf{e}_1' || \mathbf{v}_1$.

$$\mathbf{v}_1 = (2,0,0)^T, \qquad \mathbf{v}_2 = (1,-3,0)^T, \qquad \mathbf{v}_3 = (3,4,-2)^T.$$
$$\mathbf{v}_1 = (-1,0,1)^T, \qquad \mathbf{v}_2 = (1,1,3)^T, \qquad \mathbf{v}_3 = (2,-5,-4)^T.$$

Homework Problem 5: Non-orthogonal bases and metric [4]
Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[1](M)

Consider the vectors $\mathbf{v}_1 = (1,2,1)^T$, $\mathbf{v}_2 = (1,0,2)^T$, and $\mathbf{v}_3 = (2,2,0)^T$, written as column vectors in the standard basis of $\mathbb{R}^3$.

(a) Write the standard basis vector $\mathbf{e}_1 = (1, 0, 0)^T$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Ditto for $\mathbf{e}_2 = (0, 1, 0)^T$ and $\mathbf{e}_3 = (0, 0, 1)^T$. Do $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$?

(b) Let $\mathbf{x} = \mathbf{v}_1 x^1 + \mathbf{v}_2 x^2 + \mathbf{v}_3 x^3$ and $\mathbf{y} = \mathbf{v}_1 y^1 + \mathbf{v}_2 y^2 + \mathbf{v}_3 y^3$ be three vectors in $\mathbb{R}^3$, whose components w.r.t. $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are given by $x^1 = 2$, $x^2 = -5$, $x^3 = 3$ and $y^1 = 4$, $y^2 = -1$, $y^3 = -2$, respectively. Express $\mathbf{x}$ and $\mathbf{y}$ as column vectors in the standard basis of $\mathbb{R}^3$ and compute their scalar product, $\mathbf{x} \cdot \mathbf{y}$.

(c) Find the components of the metric $g_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ explicitly.

(d) Now calculate the scalar product of $\mathbf{x}$ and $\mathbf{y}$ using the formula $\mathbf{x} \cdot \mathbf{y} = x^i g_{ij} y^j = x_j y^j$, with $x_j = x^i g_{ij}$, and carry out the sum over i and j explicitly. [Check: is the result consistent with that from (b)?]

Homework Problem 6: Vector space of polynomials [3]
Points: (a)[1](E); (b)[M](E); (c)[1](E)

The goal of this problem is to show that the set of all polynomials of degree n form a vector space of dimension $n + 1$, isomorphic to $\mathbb{R}^{n+1}$.
[Remark on the notation: In the context of the present problem on polynomials, x^k means "x to the power of k", and a_k is "the coefficient of x^k". This is in contrast to the notation that we have adopted elsewhere when discussing vectors, where x^k stands for the k component of the vector $\mathbf{x} = \sum_k \mathbf{v}_k x^k$ with respect to a basis of vectors $\{\mathbf{v}_k\}$. Every notational convention has exceptions!]

Let $p_\mathbf{a}$ denote a polynomial in the variable $x \in \mathbb{R}$ of degree n:

$$p_\mathbf{a} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_\mathbf{a}(x) \equiv a_0 x^0 + a_1 x^1 + \dots a_n x^n. \tag{5}$$

$p_\mathbf{a}$ is uniquely specified by its $n+1$ real coefficients $a_0, a_1, \dots, a_n$, which for notational brevity we arrange into a $(n+1)$-tuplet, $\mathbf{a} = (a_0, a_1, \dots, a_n)^T \in \mathbb{R}^{n+1}$. Let $P_n = \{p_\mathbf{a} | \mathbf{a} \in \mathbb{R}^{n+1}\}$ denote the set of all such polynomials of degree n. The natural definitions for adding such polynomials, or multiplying them by a scalar $c \in \mathbb{R}$, is:

$$p_\mathbf{a} \boldsymbol{+} p_\mathbf{b} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto p_\mathbf{a}(x) + p_\mathbf{b}(x)\,, \tag{6}$$

$$c \boldsymbol{\cdot} p_\mathbf{a} : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto c\, p_\mathbf{a}(x)\,, \tag{7}$$

where on the right side the usual addition and multiplication in $\mathbb{R}$ is used.

(a) Show that the above addition and scalar multiplication imply the following composition rules in P_n,

Addition of polynomials:	$\boldsymbol{+}$:	$P_n \times P_n \to P_n,$	$(p_\mathbf{a}, p_\mathbf{b}) \mapsto p_\mathbf{a} \boldsymbol{+} p_\mathbf{b} \equiv p_{\mathbf{a}+\mathbf{b}}\,,$
Multiplication by a scalar:	$\boldsymbol{\cdot}$:	$\mathbb{R} \times P_n \to P_n,$	$(c, p_\mathbf{x}) \mapsto c \boldsymbol{\cdot} p_\mathbf{a} \equiv p_{c\mathbf{a}}\,,$

where $\mathbf{a} + \mathbf{b}$ and $c\mathbf{a}$ denote the usual addition and scalar multiplication in $\mathbb{R}^{n+1}$.

(b) Show that $(P_n, \boldsymbol{+}, \boldsymbol{\cdot})$ is an $\mathbb{R}$ vector space, and that it is isomorphic to $\mathbb{R}^{n+1}$.

(c) Construct a set $n + 1$ of polynomials, $\{p_{\mathbf{a}_0}, \dots, p_{\mathbf{a}_n}\} \subset P_n$, that forms a basis for this vector space.

Homework Problem 7: Inner product and norm for the vector space of continuous functions [3]
Points: (a)[2](M); (b)[1](M)

This problem illustrates a particularly important example of an inner product: in the space of continuous functions, an inner product can be defined via integration.
Let V be the vector space of *continuous* real functions defined on an interval $I \in \mathbb{R}$, $f : I \to \mathbb{R}$, with the usual composition rules of vector addition and scalar multiplication:

$$\forall f, g \in V : \qquad f + g : I \to \mathbb{R}, \qquad x \mapsto (f+g)(x) \equiv f(x) + g(x)),$$
$$\forall f \in V, \lambda \in \mathbb{R} : \qquad \lambda \cdot f : I \to \mathbb{R}, \qquad x \mapsto (\lambda \cdot f)(x) \equiv \lambda\,(f(x))\,.$$

(a) Show that the following map defines an inner product on V:

$$\langle \cdot, \cdot \rangle : V^2 \to \mathbb{R}, \qquad (f, g) \mapsto \langle f, g \rangle \equiv \int_I \mathrm{d}x\, f(x) g(x)\ .$$

(b) Now consider $I = [-1, 1]$. Compute $\langle f_1, f_2 \rangle$ for $f_1(x) \equiv \sin\left(\frac{x}{\pi}\right)$ and $f_2(x) \equiv \cos\left(\frac{x}{\pi}\right)$.

Homework Problem 8: Real vector space with unconventional composition rules [Bonus]

Points: (a)[1](M,Bonus); (b)[1](M,Bonus); (c)[1](E,Bonus)

For any $\mathbf{a} \in \mathbb{R}^2$, let $V_\mathbf{a} \equiv \{\mathbf{v}_\mathbf{x}\}$ be a set whose elements $\mathbf{v}_\mathbf{x}$, labelled by vectors $\mathbf{x} \in \mathbb{R}^2$, satisfy the following composition rules:

Addition: $\boldsymbol{+} : \quad V_\mathbf{a} \times V_\mathbf{a} \to V_\mathbf{a}, \qquad (\mathbf{v}_\mathbf{x}, \mathbf{v}_\mathbf{y}) \mapsto \mathbf{v}_\mathbf{x} \boldsymbol{+} \mathbf{v}_\mathbf{y} \equiv \mathbf{v}_{\mathbf{x}+\mathbf{y}-\mathbf{a}}$

Multiplication by a scalar: $\boldsymbol{\cdot} : \quad \mathbb{R} \times V_\mathbf{a} \to V_\mathbf{a}, \qquad (\lambda, \mathbf{v}_\mathbf{x}) \mapsto \lambda \boldsymbol{\cdot} \mathbf{v}_\mathbf{x} \equiv \mathbf{v}_{\lambda\mathbf{x}+f(\mathbf{a},\lambda)}$

Here $f(\mathbf{a}, \lambda)$ is a function that depends linearly on both $\mathbf{a}$ and λ.

(a) Show that $V_\mathbf{a}$, endowed with the composition rule $\boldsymbol{+}$, forms an abelian group, and specify the neutral element of addition and the additive inverse of $\mathbf{v}_\mathbf{x}$.

(b) Find the specific form of f that ensures that the triple $(V_\mathbf{a}, \boldsymbol{+}, \boldsymbol{\cdot})$ forms an $\mathbb{R}$-vector space.

(c) Would a similar construction work for $\mathbf{a}$, $\mathbf{x} \in \mathbb{R}^n$ (with n a positive integer) instead of $\mathbb{R}^2$?

[Total Points for Homework Problems: 21]
