Fakultät für Physik
R: Rechenmethoden für Physiker, WiSe 2015/16
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http://homepages.physik.uni-muenchen.de/~vondelft/Lehre/15r/

Sheet 03.2: Vector Product, Curves, Line Integrals

Posted: Friday, 30.10.15 Due: Friday, 06.11.15, 13:00 Central Tutorial: 11.11.15
[2](E/M/A) means: problem counts 2 points and is easy/medium hard/advanced

Example Problem 1: $1/(1-x^2)$ **Integrals by hyperbolic substitution [3]**
Points: (a)[1](E); (b)[2](M)

(a) Show that the functions $\tanh(y)$ ('hyperbolic tangent') and $\mathrm{sech}(y) = \frac{1}{\cosh(y)}$ ('hyperbolic secans') satisfy the following identities:

$$\tfrac{\mathrm{d}}{\mathrm{d}y}\tanh(y) = \mathrm{sech}^2(y), \qquad 1-\tanh^2(y) = \mathrm{sech}^2(y).$$

The second of these is useful for solving integrals that contain $1-x^2$ by using the trigonometric substitution $x = \tanh(y)$, with inverse function $y = \mathrm{artanh}(y)$, since $1-x^2 = \mathrm{sech}^2(y)$.

Calculate the following integral for $|z|<1$; check your results by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.

(b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\,\frac{1}{1-x^2}$. [Check your result: $I\left(\frac{3}{5}\right) = \ln 2$.]

Example Problem 2: Calculating with vectors [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

Given the vectors $\mathbf{a} = (4,3,1)^T$ and $\mathbf{b} = (1,-1,1)^T$.

(a) Calculate $\|\mathbf{b}\|$, $\mathbf{a}-\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

(b) Decompose $\mathbf{a}$ into a vector $\mathbf{a}_\parallel$ parallel and a vector $\mathbf{a}_\perp$ perpendicular to $\mathbf{b}$.

(c) Calculate $\mathbf{a}_\parallel\cdot\mathbf{b}$, $\mathbf{a}_\perp\cdot\mathbf{b}$, $\mathbf{a}_\parallel\times\mathbf{b}$ and $\mathbf{a}_\perp\times\mathbf{b}$. Do these results match your expectations?

Example Problem 3: Scalar triple product [2]
Points: (a)[0.5](E); (b)[1](E); (c)[0.5](E)

This problem illustrates and important relation between the scalar triple product and the question whether three vectors in $\mathbb{R}^3$ are linearly independent or not.

(a) Compute the scalar triple product $S(y)$ of $\mathbf{v}_1 = (1,0,2)^T$, $\mathbf{v}_2 = (3,2,1)^T$ and $\mathbf{v}_3 = (-1,-2,y)^T$ as function of the variable y. [Check your result: $S(1) = -4$].

(b) By solving the vector equation $\mathbf{v}_i a^i = \mathbf{0}$, find that value for y for which $\mathbf{v}_1$, $\mathbf{v}_2$ are $\mathbf{v}_3$ *not* linearly independent.

(c) What is the value of $S(y)$ for that value of y found in (b)? Interpret your result!

Example Problem 4: Grassmann identity (BAC-CAB) and Jacobi identity [5]
Points: (a)[2](M); (b)[1](E); (c)[2](M)

(a) Prove the Grassmann (or BAC-CAB) identity for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Hint: use the identity $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ for the Levi-Civita-tensor.

(b) Use the Grassmann identity to derive the Jacobi identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}\,.$$

(c) Check both identities explicitly for $\mathbf{a} = (1,1,2)^T$, $\mathbf{b} = (3,2,0)^T$ and $\mathbf{c} = (2,1,1)^T$ by separately computing all terms they contain.

Example Problem 5: Velocity and acceleration [3]
Points: (a)[1](E); (b)[1](M); (c)[1](E)

Consider the curve $\gamma = \{\mathbf{r}(t)\,|\,t \in [0, 2\pi/\omega]\}$, $\mathbf{r}(t) = (aC(t), S(t))^T \in \mathbb{R}^2$, with $C(t) = \cos\left[\pi\left(1 - \cos\omega t\right)\right]$, $S(t) = \sin\left[\pi((1 - \cos\omega t)\right]$, and $0 < a, \omega \in \mathbb{R}$.

(a) Calculate the curve's velocity vector, $\dot{\mathbf{r}}(t)$ and it's acceleration vector $\ddot{\mathbf{r}}(t)$. Can $\mathbf{r}(t)$ be expressed in terms of $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$?

(b) Can you represent the curve without the parameter t using an equation? Do you recognize the curve? Sketch the curve for the case $a = 2$.

(c) Calculate $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)$. For which values of a is $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 0$ true for all t?

Example Problem 6: Natural parametrization of a curve [2]
Points: (a)[1](E); (b)[0.5](M); (c)[0.5](E)

Consider the curve $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)^T \in \mathbb{R}^2$ for $t \in [0, 2\pi]$.

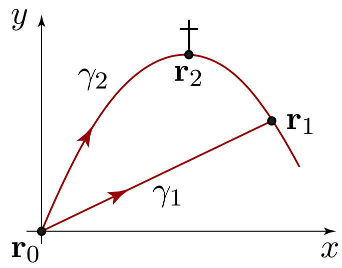
(a) Sketch the curve qualitatively.

(b) Determine it's arc length $s(t)$ in the time interval $[0, t]$. [Check your answer: $s(2\pi) = 8$.]

(c) Find the natural parametrization $\mathbf{r}_L(s)$. [Check your answer: $\mathbf{r}_L(4) = (\pi, 2)^T$.]

Example Problem 7: Line integral: mountain hike [3]
Points: [3](M)

Two hikers want to hike from point $\mathbf{r}_0 = (0,\,0)^T$ in the valley to a hut at point $\mathbf{r}_1 = (3,\,3a)^T$. Hiker 1 chooses the straight path from valley to hut, γ_1. Hiker 2 chooses a parabolic path, γ_2, via the mountain top at $\mathbf{r}_2 = (2,\,4a)^T$, the apex of the parabola (see figure). They are acted on by the force of gravity $\mathbf{F}_g = -10\,\mathbf{e}_y$, and a height-dependent wind force, $\mathbf{F}_w = -y^2\,\mathbf{e}_x$.



Find the work, $W[\gamma_i] = -\int_{\gamma_i} \mathrm{d}\mathbf{r}\cdot\mathbf{F}$, performed by the hikers along γ_1 and γ_2, as function of the parameter a. [Check your results: for $a = 1$ one finds $W[\gamma_1] = 39$, $W[\gamma_2] = 303/5$.]

[Total Points for Example Problems: 21]

Homework Problem 1: $1/(1+x^2)$ Integrals by trigonometric substitution [5]
Points: (a)[1](E); (b)[2](M); (c)[2](M)

(a) Show that the functions $\tan(y)$ and $\sec(y) = \frac{1}{\cos(y)}$ ('secans') satisfy the following identities:

$$\frac{\mathrm{d}}{\mathrm{d}y}\tan(y) = \sec^2(y), \qquad 1+\tan^2(y) = \sec^2(y).$$

The second of these is useful for solving integrals that contain $1+x^2$ by using the trigonometric substitution $x = \tan(y)$, with inverse function $y = \arctan(x)$, since $1+x^2 = \sec^2(y)$.

Calculate the following integrals; check your results by calculating $\frac{\mathrm{d}I(z)}{\mathrm{d}z}$.

(b) $I(z) = \displaystyle\int_0^z \mathrm{d}x\,\frac{1}{1+x^2}$. [Check your result: $I(\infty) = \frac{\pi}{2}$.]

(c) $I(z) = \displaystyle\int_0^z \mathrm{d}x\,\frac{1}{(1+x^2)^3}$. [Check your result: $I(1) = \frac{1}{32}(8+3\pi)$.]

Homework Problem 2: Calculating with vectors [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

Given the vectors $\mathbf{a} = (1,0,3)^T$ and $\mathbf{b} = (-5,2,1)^T$.

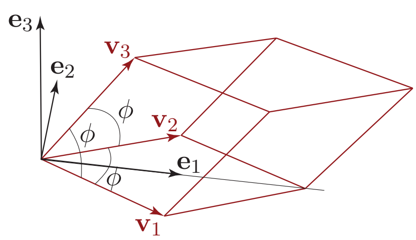
(a) Calculate $\|\mathbf{b}\|$, $\mathbf{a}-\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

(b) Decompose $\mathbf{a}$ into a vector $\mathbf{a}_\parallel$ parallel and a vector $\mathbf{a}_\perp$ perpendicular to $\mathbf{b}$.

(c) Calculate $\mathbf{a}_\parallel\cdot\mathbf{b}$, $\mathbf{a}_\perp\cdot\mathbf{b}$, $\mathbf{a}_\parallel\times\mathbf{b}$ und $\mathbf{a}_\perp\times\mathbf{b}$. Do these results match your expectations?

Homework Problem 3: Scalar triple product [3]
Points: [3](M)

Compute the volume $V(\phi)$ of the parallelepiped spanned by three unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, each pair of which encloses a mutual angle of ϕ (with $0 \le \phi \le \frac{2}{3}\pi$; why is this restriction needed?).
Check your results: (i) What do you expect for $V(\frac{\pi}{2})$ and $V(\frac{2}{3}\pi)$? (ii): $V(\frac{\pi}{3}) = \frac{1}{\sqrt{2}}$.



Hint: Choose the orientation of the parallelepiped such that $\mathbf{v}_1$ and $\mathbf{v}_2$ both lie in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, and that $\mathbf{e}_1$ bisects the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ (see figure).

Homework Problem 4: Lagrange identity [3]
Points: (a)[1](E); (b)[1](E); (c)[1](E)

(a) Prove the Lagrange identity for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^3$:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Hint: Use the properties of the Levi-Civita tensor.

(b) Use (a) to compute $\|\mathbf{a} \times \mathbf{b}\|$ and write the result in terms of $\|\mathbf{a}\|$, $\|\mathbf{b}\|$ and the angle ϕ between $\mathbf{a}$ and $\mathbf{b}$.

(c) Check the Lagrange identity explicitly for the vectors $\mathbf{a} = (1, 2, -1)^T$, $\mathbf{b} = (1, 2, 1)^T$, $\mathbf{c} = (-3, 0, 1)^T$, $\mathbf{d} = (0, 1, -2)^T$, by separately computing all its terms.

Homework Problem 5: Velocity and acceleration [2]
Points: (a)[1](E); (b)[0.5](E); (c)[0.5](E)

Consider the curve $\gamma = \{\mathbf{r}(t) \,|\, t \in [-\infty, \infty]\}$, $\mathbf{r}(t) = (\mathrm{e}^{-t^3}, a\mathrm{e}^{t^3})^T \in \mathbb{R}^2$, with $0 < a \in \mathbb{R}$.

(a) Calculate the curve's velocity vector, $\dot{\mathbf{r}}(t)$, and it's acceleration vector $\ddot{\mathbf{r}}(t)$. Can $\mathbf{r}(t)$ be expressed in terms of $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$?

(b) Can you represent the curve without the parameter t using an equation? Do you recognize the curve? Sketch the curve for the case $a = 2$.

(c) Calculate $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t)$. Find the time $t(a)$, for which $\mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 0$ holds? [Check your result: $t(\mathrm{e}^{-2}) = 1$.

Homework Problem 6: Natural parametrization of a curve [4]
Points: (a)[1](M); (b)[0.5](E); (c)[0.5](E); (d)[1](E); (e)[1](E)

Consider the curve $\gamma = \{\mathbf{r}(t) \,|\, t \in [0, \tau]\}$, $\mathbf{r}(t) = \mathrm{e}^{ct}(\cos \omega t, \sin \omega t)^T \ \in \mathbb{R}^2$, with $c \in \mathbb{R}$.

(a) Sketch the curve for the case of $\tau = 8\pi/\omega$ and $c = 1/\tau$. [This information only applies to part (a), not for parts (b-f).]

(b) Calculate the magnitude of the velocity curve, $\|\dot{\mathbf{r}}(t)\|$.

(c) Calculate the arc length $s(t)$ in the time interval $[0, t]$.

(d) Determine the natural parametrization $\mathbf{r}_L(s)$.

(e) Check explicitly that $\left\|\frac{\mathrm{d}\mathbf{r}_L}{\mathrm{d}s}\right\| = 1$.

[Check your answer: for $c = \omega = \tau = 1$: (b) $\sqrt{2}\mathrm{e}^t$, (c) $\sqrt{2}(\mathrm{e}^t - 1)$, (d) $\mathbf{r}_L(s) = [s/\sqrt{2} + 1]\left(\cos[\ln(s/\sqrt{2}+1)], \sin[\ln(s/\sqrt{2}+1)]\right)^T$.]

Homework Problem 7: Line integrals in Cartesian coordinates [4]
Points: (a)[2](M); (b)[1](E); (c)[1](M); (d)[1](M,Bonus)

Let $\mathbf{F}(\mathbf{r}) = (x^2, z, y)^T$ be a three-dimensional vector field in Cartesian coordinates, with $\mathbf{r} = (x, y, z)^T$. Calculate the line integral $\int_\gamma \mathrm{d}\mathbf{r} \cdot \mathbf{F}$ along the following paths from $\mathbf{r}_0 \equiv (0,0,0)^T$ to $\mathbf{r}_1 \equiv (0,2,-1)^T$:

(a) $\gamma_a = \gamma_1 \cup \gamma_2$ is the composite path consisting of γ_1, the straight line from $\mathbf{r}_0$ to $\mathbf{r}_2 \equiv (1,1,1)^T$, and γ_2, the straight line from $\mathbf{r}_2$ to $\mathbf{r}_1$.

(b) γ_b is parametrized by $\mathbf{r}(t) = (\sin(\pi t), 2t^{1/2}, -t^2)^T$, with $0 \le t \le 1$.

(c) γ_c is a parabola in the y-z-plane with the form $z(y) = y^2 - \frac{5}{2}y$.

Hint: the answers to (a), (b) and (c) are all the same. Optional question (d): Why is that so?

[Total Points for Homework Problems: 24]
