



Sheet 05.2: Curvilinear Coordinates

Posted: Friday, 13.11.15 Due: Friday, 20.11.15, 13:00 Central Tutorial: 25.11.15

[2](E/M/A) means: problem counts 2 points and is easy/medium hard/advanced

Example Problem 1: Surface integrals in Cartesian coordinates [2]

Points: [2](M)

Calculate the surface integral $I(a) = \int_{G_a} dx dy f(x, y)$ of the function $f(x, y) = xy$, over the area $G = \{(x, y) \in \mathbb{R}^2; 0 \leq y \leq 1; 1 \leq x \leq a - y\}$, with $2 \leq a \in \mathbb{R}$. [Check your result: $I(2) = \frac{5}{24}$].

Example Problem 2: Area calculation with Cartesian coordinates [3]

Points: (a)[1](E); (b)[1](M); (c)[1](M)

Consider the curve $\gamma_1 : \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (t, b(1 - t/a))^T$ and the closed curve $\gamma_2 : [0, 2\pi] \subset \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (a \cos t, b \sin t)^T$ in Cartesian coordinates, with $0 < a, b \in \mathbb{R}$.

- Sketch the curves γ_1 and γ_2 .
- Compute the area $S(a, b)$ enclosed by γ_2 . [Check your result: $S(1, 1) = \pi$.]
- γ_1 divides the area enclosed by γ_2 into two parts. Find the area $A(a, b)$ of the smaller part by computing an area integral. Check your result using elementary geometrical considerations.

Example Problem 3: Area integral: volume of a pyramid [3]

Points: (a)[1](E); (b)[2](M)

Consider the pyramid bounded by the xy plane, the yz plane, the xz plane and the plane $E = \{(x, y, z) \in \mathbb{R}^3, z = c(1 - x/a - y/b)\}$, with $0 < a, b, c \in \mathbb{R}$.

- Make a qualitative sketch of the pyramid. Find its volume $V(a, b, c)$ using geometric arguments [Check your result: $V(1, 1, 1) = \frac{1}{6}$.]
- Compute $V(a, b, c)$ by integrating the height $h(x, y)$ of the pyramid over its base area in the xy plane.

Example Problem 4: Coordinate calculations [3]

Points: [3](E)

Consider three points whose Cartesian coordinates (x, y, z) are $P_1: (3, -2, 4)$, $P_2: (1, 1, 1)$ and $P_3: (-3, 0, -2)$. What is the representation of these three points in cylindrical coordinates, (ρ, ϕ, z) , and in spherical coordinates, (r, θ, ϕ) ? (Give the angles in radians.)

Example Problem 5: Cylindrical coordinates: velocity, kinetic energy, angular momentum [5]

Points: (a)[2](E); (b)[1](E); (c)[1](E); (d)[0,5](E); (e)[0,5](E)

The relation between Cartesian and cylindrical coordinates is given by: $x = \rho \cos \phi$, $y = \rho \sin \phi$, $z = z$, with $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, $z \in (-\infty, \infty)$.

Basis vectors: Construct the basis vectors for cylindrical coordinates, $\{\mathbf{e}_{y_i}\} = \{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$, and show explicitly that they have the following properties:

(a) $\mathbf{e}_{y_i} \cdot \mathbf{e}_{y_j} = \delta_{ij}$ and (b) $\mathbf{e}_{y_i} \times \mathbf{e}_{y_j} = \varepsilon_{ijk} \mathbf{e}_{y_k}$.

Physical quantities: Show that in cylindrical coordinates (c) the velocity vector $\mathbf{v} = \frac{d}{dt} \mathbf{r}$, (d) the kinetic energy $T = \frac{1}{2} m \mathbf{v}^2$ and (e) the angular momentum $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ have the following forms:

$$\begin{aligned} \mathbf{v} &= \dot{\rho} \mathbf{e}_\rho + \rho \dot{\phi} \mathbf{e}_\phi + \dot{z} \mathbf{e}_z, & T &= \frac{1}{2} m [\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2], \\ \mathbf{L} &= m [-z \rho \dot{\phi} \mathbf{e}_\rho + (z \dot{\rho} - \rho \dot{z}) \mathbf{e}_\phi + \rho^2 \dot{\phi} \mathbf{e}_z]. \end{aligned}$$

Example Problem 6: Polar coordinates: line integral along spiral [2]

Points: (a)[2](E); (b)[1](E)

The curve $\gamma_S = \{\mathbf{r}(\rho, \phi) \in \mathbb{R}^2 \mid \rho = R + \frac{1}{2\pi} \phi \Delta, \phi \in [0, 2\pi]\}$, with $0 < R, \Delta \in \mathbb{R}$, describes a spiral path in two dimensions, parametrised using polar coordinates.

(a) Sketch the spiral path γ_S and calculate the line integral $W_1[\gamma_S] = \int_{\gamma_S} d\mathbf{r} \cdot \mathbf{F}_1$ of the field $\mathbf{F}_1 = \mathbf{e}_\phi$ along γ_S . [Check your result: if $R = \Delta = 1$, then $W_1[\gamma] = 3\pi$.]

(b) Calculate the line integral $W_2[\gamma] = \int_\gamma d\mathbf{r} \cdot \mathbf{F}_2$ of the field $\mathbf{F}_2 = \mathbf{e}_x$ along the straight path γ_G from the point $(R, 0)^T$ to the point $(R + \Delta, 0)^T$, and also along the spiral path γ_S . Are the results related? Explain!

Example Problem 7: Line integral in spherical coordinates: satellite in orbit [5]

Points: (a)[1,5](E); (b)[1](E); (c)[0,5](E); (d)[2](M)

A satellite travels along an orbit γ that encircles the north-south-axis from a point over the north pole to a point over the south pole. In cylindrical coordinates the orbit is given by $r(t) = r_0$, $\theta(t) = \omega_1 t$, $\phi(t) = \omega_2 t$, with $t \in [0, \pi/\omega_1]$. Due to the rotation of the earth, there is a wind in the upper atmosphere which exerts a force $\mathbf{F} = -F_0 \sin \theta \mathbf{e}_\phi$ on the satellite.

(a) Make a qualitative sketch of the orbit, for $\omega_2 = 20\omega_1$. How many times does the path wind around the north-south axis?

(b) What is velocity vector $\dot{\mathbf{r}}$ written in spherical coordinates?

(c) Give the length $L[\gamma]$ of the orbit in terms of an integral. (You are not required to solve it.)

(d) Calculate, using the line integral $W[\gamma] = \int_\gamma d\mathbf{r} \cdot \mathbf{F}$, the work performed against the wind by the satellite along its orbit. [Check your result: if $F_0 = r_0 = \omega_1 = \omega_2 = 1$, then $W[\gamma] = -\frac{\pi}{2}$.]

[Total Points for Example Problems: 23]

Homework Problem 1: Surface integrals in Cartesian coordinates [2]

Points: [2](M)

Calculate the surface integral $I(a) = \int_G dx dy f(x, y)$ of the function $f(x, y) = y^2 + x^2$ over the surface $G = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 1; 0 \leq y \leq e^{ax}\}$, with $a \in \mathbb{R}$. *Hint:* for $\int dx x^2 e^{ax}$, use partial integration twice! [Check your result: $I(1) = e + (e^3 - 19)/9$.]

Homework Problem 2: Area calculation with Cartesian coordinates [3]

Points: (a)[1](E); (b)[2](M)

Consider the curves $\gamma_1 : \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto ((t - 2a)^2 + 2a^2, t)^T$ and $\gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (2(t - a)^2, t)^T$ in Cartesian coordinates, with $0 < a \in \mathbb{R}$.

(a) Sketch the curves γ_1 and γ_2 .

(b) Compute the finite area $S(a)$ enclosed between these curves. [Check your result: $S(\frac{1}{2}) = \frac{4}{3}$.]

Homework Problem 3: Surface integrals: Volume of an ellipsoidal tent [3]

Points: (a)[1](E); (b)[2](M)

A tent has a flat, ellipsoidal base, given by the equation $(x/a)^2 + (y/b)^2 \leq 1$. The shape of the tent's roof is given by the height function $h(x, y) = c[1 - (x/a)^2 - (y/b)^2]$.

(a) Give a qualitative sketch of the shape of the tent, for $a = 2$, $b = 1$ and $c = 2$.

(b) Calculate the volume V of the tent via a surface integral of the height function. [Check your result: if $a = b = c = 1$, then $V = \pi/2$.]

Hint: Show by a suitable trigonometric substitution that $\int_0^1 dx (1 - x^2)^{3/2} = \frac{3}{16}\pi$.

Homework Problem 4: Coordinate transformations [2]

Points: [2](E)

The point P_1 has spherical coordinates $(r, \theta, \phi) = (2, \pi/4, 3\pi/4)$. What are its cartesian and cylindrical coordinates, (x, y, z) and (ρ, ϕ, z) , respectively? The point P_2 has cylindrical coordinates $(\rho, \phi, z) = (4, \pi/2, 1)$. What are its cartesian and spherical coordinates? (Give the angles in radians.)

Homework Problem 5: Cylindrical coordinates: velocity, kinetic energy, angular momentum [5]

Points: (a)[2](E); (b)[1](E); (c)[1](E); (d)[0,5](E); (e)[0,5](E)

The relationship between Cartesian and spherical coordinates is given by: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, with $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$.

Basis vectors: Construct the basis vectors for spherical coordinates, $\{\mathbf{e}_{y_i}\} = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$, and show explicitly that

(a) $\mathbf{e}_{y_i} \cdot \mathbf{e}_{y_j} = \delta_{ij}$ und (b) $\mathbf{e}_{y_i} \times \mathbf{e}_{y_j} = \varepsilon_{ijk} \mathbf{e}_{y_k}$.

Physical quantities: Show that in spherical coordinates (c) the velocity vector $\mathbf{v} = \frac{d}{dt}\mathbf{r}$, (d) the kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$ and (e) the angular momentum $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ are as follows:

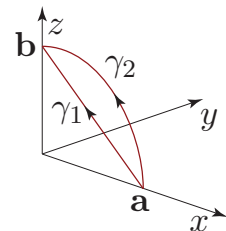
$$\mathbf{v} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi, \quad T = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta], \quad \mathbf{L} = mr^2[\dot{\theta} \mathbf{e}_\phi - \dot{\phi} \sin \theta \mathbf{e}_\theta].$$

Homework Problem 6: Line integral in Cartesian and spherical coordinates [3]

Points: (a)[1](E); (b)[1](E); (c)[1](E)

Consider the vector field $\mathbf{F} = f(0, 0, z)^T$. Compute the line integral $W[\gamma] = \int_\gamma d\mathbf{r} \cdot \mathbf{F}$ from $\mathbf{a} = (1, 0, 0)^T$ to $\mathbf{b} = (0, 0, 1)^T$ explicitly along the following two paths:

- (a) γ_1 : a straight line. [Check your result: if $f = 2$, then $W[\gamma_1] = 1$.]
 (b) γ_2 : a segment of a circle with radius $R = 1$ centered at the origin. Use spherical coordinates. [Check your result: for $f = 3$, we have $W[\gamma_2] = \frac{3}{2}$.]
 (c) How are the results from (a) and (b) related? Explain!



Homework Problem 7: Line integrals in cylindrical coordinates: bathtub drain [5]

Points: (a)[1](E); (b)[1](E); (c)[1](M); (d)[2](M)

A soap bubble travels along a spiral-shaped path γ towards the drain of a bathtub. In cylindrical coordinates the path is given by $\rho(t) = \rho_0 e^{-t/\tau}$, $\phi(t) = \omega t$, $z(t) = z_0 e^{-t/\tau}$, with $\rho_0 > \rho_d$ and $t \in [0, t_d]$, where ρ_d is the drain radius and $t_d = \tau \ln(\rho_0/\rho_d)$ the time at which the bubble reaches the drain.

- (a) Make a qualitative sketch of the path (e.g. for $\omega = 6\pi/\tau$ in $\rho_0 = 10\rho_d$).
 (b) What is the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ in cylindrical coordinates? What is the magnitude of the final velocity, i.e $v_d = \|\mathbf{v}(t_d)\|$?
 (c) Show that the length of the path is given by $L[\gamma] = \tau v_d (\rho_0/\rho_d - 1)$.
 (d) Using the line integral $W[\gamma] = \int_{\gamma} d\mathbf{r} \cdot \mathbf{F}$, find the work done by gravity $\mathbf{F} = -mg\mathbf{e}_z$ along the path of the soap bubble. Give a physical interpretation for this result!

[Check your results: if $\tau = 2/\omega$, $z_0 = 2\rho_0$ and $\rho_d = \rho_0/3$, then: (b) $v_d = \rho_0/\tau$, (c) $L = 2\rho_0$, (d) $W[\gamma] = mg\rho_0 4/3$.]

[Total Points for Homework Problems: 23]
