Fakultät für Physik
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Sheet 09.3: Series Expansions

Posted: Friday, 11.12.15 Due: Friday, 18.12.15, 13:00 Central Tutorial: 23.12.15
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Example problems: {T}: will be discussed in tutorial; {S}: self study.

Example Problem 1: Addition theorems for sine and cosine [1]
Points: (a)[0.5](E); (b)[0.5](E). (S)

Prove the addition theorems for sine and cosine:

(a) $\cos(a+b) = \cos a \cos b - \sin a \sin b$,

(b) $\sin(a+b) = \cos a \sin b + \sin a \cos b$.

Hint: Use the Euler-de Moivre identity on both sides of $\mathrm{e}^{\mathrm{i}(a+b)} = \mathrm{e}^{\mathrm{i}a}\mathrm{e}^{\mathrm{i}b}$.

Example Problem 2: Taylor series [3]
Points: (a)[1](E); (b)[1](E); (c)[1](M). [T]

Find the Taylor series of the following functions. You may choose to either calculate the coefficients of the Taylor series by taking the corresponding derivatives, or to use the known Taylor expansions of $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$ and $\ln(1+x)$.

(a) $f(x) = \frac{1}{1-\sin(x)}$ around $x = 0$, up to and including fourth order.

(b) $g(x) = \sin(\ln(x))$ around $x = 1$, up to and including second order.

(c) $h(x) = \mathrm{e}^{\cos x}$ around $x = 0$, up to and including second order.

Check your results: the highest-order term requested in each case is: (a) $\frac{2}{3}x^4$, (b) $-\frac{1}{2}(x-1)^2$, (c) $-\mathrm{e}\frac{1}{2}x^2$.

Example Problem 3: Series expansion for iteratively solving an equation [2]
Points: (a)[1](M); (b)[1](M). [T]

(a) Solve the quadratic equation $y^2 - 1 = 2\varepsilon y$ up to and including $\mathcal{O}(\varepsilon^2)$ for small ε, i.e. express y in the form $y = y_0 + y_1\varepsilon + \frac{1}{2!}y_2\varepsilon^2 + \mathcal{O}(\varepsilon^3)$. *Hint:* Note that the equation can have more than one solution. [Check your results: $y_2 = \pm 1$.]

(b) Next, find the exact solutions of this equation, and calculate the first three terms of their Taylor series. Check that these expansions match those obtained from the iterative solution.

Example Problem 4: Taylor series for inverse function [2]
Points: (a)[1](M); (b)[1](M). [S]

Learning objective: Calculate the series expansion of an inverse function by iteratively solving an equation.
The inverse $g(x)$ of the function $f(x)$ fulfills the defining equation $f(g(x)) = x$. The series expansion of the inverse function around the point x_0, of the form $g(x_0+x) \equiv y(x) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0) x^n$, can be determined by iteratively solving the equation $f(y(x)) = x_0 + x$ for $y(x)$. In this manner, calculate the series expansion of the following functions around $x = 0$, up to and incuding the second order in x:

(a) $\ln(1 + x)$, (b) 2^x.

[Check your results: the highest-order term requested in each case is: (a) $-\frac{1}{2}x^2$, (b) $\frac{1}{2}\ln^2(2)x^2$.]

Example Problem 5: Taylor expansions in two dimensions [2]
Points: (a)[1](E); (b)[1](M). [S]

Find the Taylor expansion of the function $g(x, y) = \mathrm{e}^x \cos(x + 2y)$ in x and y, around the point $(x, y) = (0, 0)$. Calculate explicitly all terms up to and including second order,

(a) by multiplying out the series expansions for the exponential and cosine functions;

(b) by using the formula for the Taylor series of a function of two variables.

[Check your results: the mixed second-order term in each case is: (a) $-2xy$, (b) $-2xy$.]

Example Problem 6: Lagrange multipliers [2]
Points: [2](M). [T]

Find the extremum of the function $j(\mathbf{r}) = x^2 + y^2 + z^2$ subject to the constraints $x + y + z = 1$ and $x - y + 2z = 2$.

Example Problem 7: $1/(1 + x^2)$ integral via partial fraction decomposition [3]
Points: (a)[2](A); (b)[1](M). [S]

(a) Compute the integral $I(z) = \displaystyle\int_0^z \mathrm{d}x \frac{1}{1 + x^2}$ using a partial fraction decomposition.

(b) Alternatively, this integral can also be computed using the trigonometric substitution $y = \tan(x)$, resulting in $I(z) = \arctan(z)$. Show explicitly that your result from (a) agrees with this.

Example Problem 8: Functions of matrices [4]
Points: (a)[0.5](E); (b)[1](E); (c)[1](M); (d)[1.5](M). [T]

The purpose of this problem is to gain familiarity with the concept of a "function of a matrix". Let f be an analytic function, with Taylor series $f(x) = \sum_{l=0}^{\infty} c_l x^l$, and $A \in \mathrm{mat}(\mathbb{R}, n, n)$ a square matrix, then $f(A)$ is defined as $f(A) = \sum_{l=0}^{\infty} c_l A^l$, with $A^0 = \mathbb{1}$.

(a) A matrix A is called 'nilpotent' if an $l \in \mathbb{N}$ exists such that $A^l = 0$. Then the Taylor series of $f(A)$ ends after l terms. Example: Compute e^A for $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.

(b) If $A^2 \propto \mathbb{1}$, then $A^{2m} \propto \mathbb{1}$ and $A^{2m+1} \propto A$, and the Taylor series for $f(A)$ has the form $f_0\mathbb{1} + f_1 A$. Example: Compute e^A explicitly for $A = \theta\tilde{\sigma}$, with $\tilde{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

[Check your result: if $\theta = -\frac{\pi}{6}$, then $\mathrm{e}^A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$.]

(c) If A is diagonalizable, then $f(A)$ can be expressed in terms of its eigenvalues. Let S be the similarity transformation that diagonalizes A, with diagonal matrix $D = S^{-1}AS$ and diagonal elements $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Show that then the following relations hold:

$$f(A) = Sf(D)S^{-1} = S \begin{pmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f(\lambda_n) \end{pmatrix} S^{-1} .$$

Remark: Both equalities are to be established independently of each other.

(d) Now compute the matrix function e^A from (b) using diagonalization, as in (c).

Example Problem 9: Exponential representation of 2-dimensional rotation matrix [1]
Points: (a)[0.5](E); (b)[0.5](E). [T]

The matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ describes a rotation by the angle θ in $\mathbb{R}^2$. Use the following 'infinite product decomposition' to find an exponential representation of this matrix:

(a) A rotation by the angle θ can be represented as a sequence of m rotations, each by the angle θ/m: $R_\theta = [R_{(\theta/m)}]^m$. For $m \to \infty$ we have $\theta/m \to 0$, thus the matrix $R_{(\theta/m)}$ can be written as $R_{(\theta/m)} = \mathbb{1} + (\theta/m)\tilde{\sigma} + \mathcal{O}\big((\theta/m)^2\big)$ Find the matrix $\tilde{\sigma}$.

(b) Now use the identity $\lim_{m\to\infty}[1 + x/m]^m = \mathrm{e}^x$ to show that $R_\theta = \mathrm{e}^{\theta\tilde{\sigma}}$.

Remark: Justification of this identity: We have $\mathrm{e}^x = [\mathrm{e}^{x/m}]^m = [1 + x/m + \mathcal{O}\big((x/m)^2\big)]^m$. In the limit $m \to \infty$ the terms of order $\mathcal{O}\big((x/m)^2\big)$ can be neglected.

[Check your result: does the Taylor series for $\mathrm{e}^{\theta\tilde{\sigma}}$ reproduce the matrix for R_θ given above?]

Remark: The procedure illustrated here, by which an infinite sequence of identical, infinitesimal transformations is exponentiated, is a cornerstone of the theory of 'Lie groups', whose elements are associated with continuous parameters (here the angle θ). In this context the above matrix $\tilde{\sigma}$ is called the 'generator' of the rotation.

[Total Points for Example Problems: 20]

Homework Problem 1: Powers of Sine and Cosine [1]
Points: (a)[0.5](E); (b)[0.5](E)

Use the Euler-de Moivre identity to prove the following identities:

(a) $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos(2a)\,, \qquad \sin^2 a = \frac{1}{2} - \frac{1}{2}\cos(2a)\,.$

(b) $\cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos(3a)\,, \qquad \sin^3 a = \frac{3}{4}\sin a - \frac{1}{4}\sin(3a)\,.$

Homework Problem 2: Taylor series [3]
Points: (a)[1](E); (b)[1](E); (c)[1](M)

Taylor expand the following functions. You may choose to either calculate the coefficients of the Taylor series by taking the corresponding derivatives, or to use the known Taylor expansions of $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$ and $\ln(1+x)$.

(a) $f(x) = \frac{\cos(x)}{1-x}$ around $x = 0$. Keep all terms up to and including third order.

(b) $g(x) = \mathrm{e}^{\cos(x^2+x)}$ about $x = 0$, up to and including third order.

(c) $h(x) = \mathrm{e}^{-x}\ln(x)$ around $x = 1$, up to and including third order.

[Check your results: the highest-order term requested in each case is: (a) $\frac{1}{2}x^3$, (b) $-\mathrm{e}\,x^3$, (c) $\frac{4}{3}\mathrm{e}^{-1}(x-1)^3$.]

Homework Problem 3: Series expansion for iteratively solving an equation [2]
Points: [2](M)

A real and analytic function $f(x)$ satisfies the following equation, for $|x| \ll 1$:

$$\ln\left[(x+1)^2\right] + \mathrm{e}^{y(x)} = 1 - y(x) \; .$$

Determine $y(x)$ iteratively up to order $\mathcal{O}(x^2)$, using a series expansion of the form $y(x) = y_0 + y_1 x + \frac{1}{2!}y_2 x^2 + \mathcal{O}(x^3)$. *Hint:* Start by showing that the solution has the property $y(0) = 0$. [Check your results: $y_2 = \frac{1}{2}$.]

Homework Problem 4: Taylor series for inverse function [2]
Points: (a)[1](M); (b)[1](M)

Calculate the series expansion of $\arcsin(x)$ around $x = 0$, up to and including order three, using the following two alternative methods:

(a) Find $\arcsin(x) \equiv y(x)$ by iteratively solving the equation $\sin[y(x)] = x$.

(b) Starting from the identity arcsin $(\sin(y)) = y$, use the known series expansion for $\sin(y)$ as well as the Ansatz $\arcsin(x) = c_1 x^1 + c_3 x^3 + \mathcal{O}(x^5)$, and determine c_1 and c_3 by a comparison of coefficients. [Why are there only odd powers of x?]

Learning objective: realizing that some approaches may be easier than others!
[Check your results: $c_3 = \frac{1}{6}$.]

Homework Problem 5: Taylor expansion in two dimensions [2]
Points: (a)[0.5](E); (b)[1.5](M)

For the following functions, calculate the Taylor expansion in x and y around the point $(x, y) = (0, 0)$, up to and including second order:

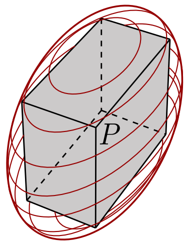
(a) $f(x,y) = e^{-(x+y)^2}$, (b) $g(x,y) = \dfrac{1+x}{\sqrt{1+xy}}$.

[Check your results: the mixed second-order term in each case is: (a) $-2xy$, (b) $-\frac{1}{2}xy$.]

Homework Problem 6: Lagrange multipliers [2]
Points: (a)[1](M); (b)[1](M)

(a) A manufacturer would like to pack his product in a rectangular box using as little material as possible, by minimizing the box' surface area A for a given volume V. Find the side lenghts x, y and z and the minimal surface area A_{min} of the box in terms of V, by solving an appropriate extremization problem. [Check your result: if $V = \frac{1}{8}\text{m}^3$ then $A = \frac{3}{2}\text{m}^2$.]

(b) Consider the ellipsoid defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Also consider a rectangular box whose corners lie on the surface of the ellipsoid and whose edges are parallel to the elipsoid's symmetry axes. Let $P = (x_p,\, y_p,\, z_p)^T$ denote that corner of the box that lies in the positive quadrant ($x_p > 0,\ y_p > 0,\ z_p > 0$). How should this corner by chosen to maximize the volume of the box? What is the value of the maximal volume?



Hint: Maximize the volume $V(x, y, z) = 8\,xyz$ of a box having a corner at $(x, y, z)^T$, under the constraint that this point lies on the ellipsoid.
[Check your result: if $a = \frac{1}{2}$, $b = 3$, $c = \sqrt{3}$, then $V_{\text{max}} = 4$.]

Homework Problem 7: $1/(1-x^2)$ integral via partial fraction decomposition [3]
Points: (a)[2](A); (b)[1](M)

(a) Compute the integral $I(z) = \displaystyle\int_0^z \mathrm{d}x \frac{1}{1-x^2}$ using a partial fraction decomposition.

(b) Alternatively, this integral can also be computed using the trigonometric substitution $y = \tanh(x)$, resulting in $I(z) = \text{artanh}(z)$. Show explicitly that your result from (a) agrees with this.

Homework Problem 8: Functions of matrices [3]
Points: (a)[0.5](E); (b)[1](E); (c)[1.5](M); (d)[1](A,Bonus).

Express each of the following matrix functions explicitly in terms of a matrix:

(a) e^A, with $A = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$.

(b) e^B, with $B = b\sigma_1$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, using the Taylor series of the exponential function.

[Check your result: if $b = \ln 2$, then $\mathrm{e}^A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.]

(c) The same function as in (b), now by diagonalizing B.

(d) e^C, with $C = \mathrm{i}\theta\,\Omega$, where $\Omega = n_j S_j$, while $\mathbf{n} = (n_1, n_2, n_3)^T$ is a unit vector ($\|\mathbf{n}\| = 1$) and S_j are the spin-$\frac{1}{2}$ matrices: $S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Hint: Start by computing Ω^2 (for this, the property $S_i S_j + S_j S_i = \frac{1}{2}\delta_{ij}\mathbb{1}$ of the spin-$\frac{1}{2}$ matrices is useful), and then use the Taylor series of the exponential function.
[Check your result: if $\theta = -\pi/2$ and $n_1 = -n_2 = n_3 = \frac{1}{\sqrt{3}}$, then $\mathrm{e}^C = \frac{1}{\sqrt{6}}\begin{pmatrix} 1-\mathrm{i} & 1-\mathrm{i} \\ -1-\mathrm{i} & 1+\mathrm{i} \end{pmatrix}$.]

Remark: The exponential form e^C is a representation of SU(2) transformations, the group of all special unitary transformations in $\mathbb{C}^2$. Its elements are characterized by three continuous real parameters (here θ, n_1 and n_2, with $n_3 = \sqrt{1 - n_1^2 - n_2^2}$). The S_j matrices are 'generators' of these transformations; they satisfy the SU(2) algebra, i.e. their commutators yield $[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k$.

Homework Problem 9: Exponential representation 3-dimensional rotation matrix [3]
Points: (a)[0.5](E); (b)[0.5](M); (c)[1](M); (d)[1](A)

In $\mathbb{R}^3$, a rotation by an angle θ, about an axis whose direction is given by the unit vector $\mathbf{n} = (n_1, n_2, n_3)$, is represented by a 3×3 matrix that has the following matrix elements:

$$(R_\theta(\mathbf{n}))_{ij} = \delta_{ij} \cos\theta + n_i n_j (1 - \cos\theta) - \epsilon_{ijk}\, n_k \sin\theta \qquad (\epsilon_{ijk} = \text{Levi-Civita-Tensor})\,. \tag{1}$$

The goal of the following steps is to supply a justification for Eq. (1).

(a) Consider first the three matrices $R_\theta(\mathbf{e}_j)$ for rotations by the angle θ about the three coordinate axes $\mathbf{e}_j$, with $j = 1, 2, 3$. Elementary geometrical considerations yield:

$$R_\theta(\mathbf{e}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_\theta(\mathbf{e}_2) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_\theta(\mathbf{e}_3) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For each of these matrices, use an infinite product decomposition of the form $R_\theta(\mathbf{n}) = \lim_{m\to\infty}[R_{\theta/m}(\mathbf{n})]^m$ to obtain an exponential representation of the form $R_\theta(\mathbf{e}_i) = \mathrm{e}^{\theta\tau_i}$. Find the three 3×3 matrices τ_1, τ_2 and τ_3. [Check your results: The τ_i commutators yield $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$. This is the so-called SO(3) algebra, which underlies the representation theory of 3-dimensional rotations. Moreover, $\tau_1^2 + \tau_2^2 + \tau_3^2 = -2\mathbb{1}$.]

(b) Now consider a rotation by the angle θ about an arbitrary axis $\mathbf{n}$. To find an exponential representation for it using an infinite product decomposition, we need an approximation for $R_{\theta/m}(\mathbf{n})$ up to first order in the small angle θ/m. It has the following form:

$$R_{(\theta/m)}(\mathbf{n}) = R_{(n_1\theta/m)}(\mathbf{e}_1) R_{(n_2\theta/m)}(\mathbf{e}_2) R_{(n_3\theta/m)}(\mathbf{e}_3) + \mathcal{O}\big((\theta/m)^2\big). \tag{2}$$

Intuitive justification: If the rotation angle θ/m is sufficiently small, the rotation can be performed in three substeps, each about the direction $\mathbf{e}_j$, by the 'partial' angle $n_j\theta/m$. The prefactors n_j ensure that for $\mathbf{n} = \mathbf{e}_j$ (rotation about a coordinate axis j) only *one* of the three factors in (2) is different from $\mathbb{1}$, namely the one that yields $R_{(\theta/m)}(\mathbf{e}_j)$; for example, for $\mathbf{n} = \mathbf{e}_2 = (0,1,0)^T$: $R_{(0\theta/m)}(\mathbf{e}_1) R_{(1n_2\theta/m)}(\mathbf{e}_2) R_{(0\theta/m)}(\mathbf{e}_3) = R_{(n_2\theta/m)}(\mathbf{e}_2)$.

Show that such a product decomposition of $R_\theta(\mathbf{n})$ yields the following exponential representation:

$$R_\theta(\mathbf{n}) = \mathrm{e}^{\theta\Omega}, \qquad \Omega = n_i\tau_i = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \qquad (\Omega)_{ij} = -\epsilon_{ijk} n_k\,. \tag{3}$$

(c) Show that Ω, the 'generator' of the rotation, has the following properties:

$$(\Omega^2)_{ij} = n_i n_j - \delta_{ij}, \qquad \Omega^l = -\Omega^{l-2} \quad \text{für } 3 \le l \in \mathbb{N}. \qquad \text{[Cayley-Hamilton theorem]} \tag{4}$$

Hint: First compute Ω^2 and Ω^3, then the form of $\Omega^{l>3}$ will be obvious.

(d) Show that the Taylor expansion of $R_\theta(\mathbf{n}) = \mathrm{e}^{\theta\Omega}$ yields the following expression,

$$R_\theta(\mathbf{n}) = \mathbb{1} + \Omega\sin\theta + \Omega^2(1 - \cos\theta), \tag{5}$$

and that its matrix elements correspond to Eq. (1).

[Total Points for Homework Problems: 21]
