Sheet 10.4: Differential Equations

Posted: Friday, 18.12.15 Due: Friday, 08.01.16, 13:00 Central Tutorial: 13.01.16
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Example problems: {T}: will be discussed in tutorial; {S}: self study.

Example Problem 1: Separation of variables [2]
Points: (a)[1](E); (b)[1](E). [T]

A differential equation of order 1 is called 'autonomous' if it has the form $\dot{x} = f(x)$, i.e. the right hand side is time independent [non-autonomous equations have $\dot{x} = f(x,t)$]. Such an equation can the solved by separation of variables.

a) Solve the autonomous differential equation $\dot{x} = x^2$ for two different initial conditions: (i) $x(0) = 1$ and (ii) $x(2) = -1$. [Check your results: (i) $x(-2) = \frac{1}{3}$, and (ii) $x(2) = -1$.]

b) Sketch the obtained solutions qualitatively. Convince yourself using a graphical analysis that the sought function $x(t)$ and its derivative $\dot{x}(t)$ satisfy the relation specified by the differential equation.

Example Problem 2: Barometric formula [1]
Points: [1](E). [S]

The standard barometric formula for atmospheric pressure $p(x)$ as a function of x is given by: $\frac{\mathrm{d}p(x)}{\mathrm{d}x} = -\alpha \frac{p(x)}{T(x)}$. Solve this equation with initial value $p(x_0) = p_0$ for the case of a linear temperature gradient, $T(x) = T_0 - b(x - x_0)$. *Hint:* Separation of variables!
[Check your result: if $\alpha, b, T_0, x_0, p_0 = 1$, then $p(1) = 1$.]

Example Problem 3: Differential equation: substitution and separation of variables [2]
Points: (a)[1](E); (b)[1](E). [T]

(a) Show that the differential equation $y' = f(y/x)$ for the function $y(x)$ can be converted by the substitution $y = ux$ into a differential equation for the function $u(x)$, which is solvable using separation of variables.

(b) Use this method to solve the equation $xy' = 2y + x$ with the initial condition $y(1) = 0$. [Check your result: $y(2) = 2$.]

Example Problem 4: Inhomogeneous differential equation [3]
Points: (a)[1](E); (b)[2](M). [T]

Solve the inhomogeneous differential equation $\dot{x} + 2x = t$ with $x(0) = 0$, as follows:

(a) Determine the general solution of the homogeneous equation.

(b) Then find a special (particular) solution to the inhomogeneous problem by means of variation of constants. [Check your result: $x(-\ln 2) = \frac{3}{4} - \frac{1}{2}\ln 2$.]

Example Problem 5: Driven overdamped harmonic oscillator [7]
Points: (a)[1](E); (b)[2](M); (c)[2](M); (d)[2](M). [T]

Consider the following driven, over-damped harmonic oscillator with $\gamma > \Omega$:

Differential equation: $$\ddot{x} + 2\gamma\dot{x} + \Omega^2 x = f_{\rm A}(t). \tag{1}$$

Initial value: $$x(0) = 0, \quad \dot{x}(0) = 1\,, \tag{2}$$

Driving function: $$f_{\rm A}(t) = \begin{cases} f_{\rm A} & \text{for} \quad t \geq 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

For $t > 0$, find a solution to this equation of the form $x(t) = x_{\rm h}(t) + x_{\rm p}(t)$, where $x_{\rm h}(t)$ and $x_{\rm p}(t)$ are the homogeneous and particular solutions to the homogeneous and inhomogeneous differential equation that have the initial values (2) or $x_{\rm p}(0) = \dot{x}_{\rm p}(0) = 0$, respectively. Proceed as follows:

(a) Rewrite as matrix equation: Write the DEQ (1) in the matrix form

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} + \mathbf{b}(t), \quad \text{with} \quad \mathbf{x} \equiv (x, \dot{x})^T \equiv (x^1, x^2)^T\,. \tag{3}$$

Find the matrix A, the driving force vector $\mathbf{b}(t)$, and the initial value $\mathbf{x}_0 = \mathbf{x}(0)$.

(b) Homogeneous solution: Find the solution $\mathbf{x}_{\rm h}(t)$ of the homogeneous DEQ $(3)|_{\mathbf{b}(t)=0}$ that has the initial value $\mathbf{x}_{\rm h}(0) = \mathbf{x}_0$. Use the Ansatz $\mathbf{x}_{\rm h}(t) = \sum_j c_{\rm h}^j \mathbf{x}_j(t)$, with $\mathbf{x}_j(t) = \mathbf{v}_j e^{\lambda_j t}$, where λ_j and $\mathbf{v}_j$ $(j = 1, 2)$ are the eigenvalues and the eigenvectors of A. What does the corresponding solution $x_{\rm h}(t) = x^1_{\rm h}(t)$ of the homogeneous differential equation $(1)|_{f_{\rm A}(t)=0}$ look like? [Check your result: if $\gamma = \sqrt{2}\ln 2$ and $\Omega = \ln 2$, then $x_h(1) = \frac{3}{4}\frac{2-\sqrt{2}}{\ln 2}$.]

(c) Particular solution: Using the Ansatz $\mathbf{x}_{\rm p}(t) = \sum_j c_{\rm p}^j(t)\mathbf{x}_j(t)$ (variation of constants), find the particular solution for the inhomogeneous differential equation (3) that has the initial value $\mathbf{x}_{\rm p}(0) = \mathbf{0}$. What is the corresponding solution $x_{\rm p}(t) = x^1_{\rm p}(t)$ of the inhomogeneous DEQ (1)? [Check your result: if $\gamma = 3\ln 2$, $\Omega = \sqrt{5}\ln 2$ and $f_A = 1$, then $x_p(1) = \frac{49}{640}\frac{1}{(\ln 2)^2}$.]

(d) Qualitative discussion: The desired solution of the inhomogeneous DEQ (1) is given by $x(t) = x_{\rm h}(t) + x_{\rm p}(t)$. Sketch your result for this function qualitatively for the case $f_{\rm A} < 0$, and explain the behavior as $t \to 0$ and $t \to \infty$.

Example Problem 6: Critically damped harmonic oscillator [6]
Points: (a)[1](E); (b)[2](M); (c)[1](E); (d)[2](M,Bonus). [T]

Find the general solution for the damped, homogeneous, harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \Omega^2 x = 0\,,$$

for the critically damped case $\gamma = \Omega$, by finding a matrix differential equation of order 1 and solving the corresponding eigenvalue problem.

(a) In this case, both the eigenvalues are degenerate and there is only one corresponding eigenvector. Find the corresponding solution $x_1(t)$.

(b) Find a second solution via variation of constants by inserting the ansatz $x_2(t) = c(t)x_1(t)$ into the DEQ for x. Find a differential equation for $c(t)$ and solve this equation.

(c) Find the solution $x(t)$ that satisfies the initial value $x(0) = 1$, $\dot{x}(1) = 1$.
[Check your result: if $\gamma = 2$, then $x(\ln 2) = \frac{1}{4}\left(1 - \ln 2(2 + e^2)\right)$.]

(d) The critically damped harmonic oscillator can be thought of as the limit $\lambda \to \Omega$ of both the over-damped (see example problem) and under-damped (see lecture notes) harmonic oscillator. Perform a Taylor expansion of the general solution of both the over-damped and under-damped cases for small values of ϵt, with $\epsilon \equiv \sqrt{|\gamma^2 - \Omega^2|}$, and show that the result in both cases can be written as a linear combination of the solutions to the critically damped harmonic oscillator found in (a) and (b).

Example Problem 7: Integration by partial fraction expansion [4]
Points: (a)[2](M); (b)[2](M). [T]

Use partial fraction expansions to compute the following integrals, for $z \in \mathbb{R}$, $z > -1$:

(a) $I(z) = \int_0^z \mathrm{d}x \, \frac{3x+3}{(x+1)^2(x-2)}$, (b) $I(z) = \int_0^z \mathrm{d}x \, \frac{3x}{(x+1)^2(x-2)}$.

[Check your results: (a) $I(3) = -\ln 8$, (b) $I(3) = -\ln 4 + \frac{3}{4}$.]

[Total Points for Example Problems: 25]

Homework Problem 1: Separation of variables [2]
Points: (a)[1](E); (b)[1](E)

(a) Solve the differential equation $y' = -x^2/y^3$ for the function $y : \mathbb{R} \to \mathbb{R}$, $x \mapsto y(x)$ by separating variables, for two different initial conditions: (i) $y(0) = 1$, and (ii) $y(0) = -1$.
[Check your result: (i) $y(-1) = \left(\frac{7}{3}\right)^{1/4}$, (ii) $y(-1) = -\left(\frac{7}{3}\right)^{1/4}$.]

(b) Sketch the obtained solution qualitatively. Convince yourself using a graphical analysis that the sought function $y(x)$ and its derivative $y'(x)$ satisfy the relation specified by the differential equation.

Homework Problem 2: Bacterial culture with toxin [4]
Points: (a)[1](E); (b)[1](M); (c)[1](E); (d)[1](E)

A bacterial culture is exposed to the effects of a toxin. The death rate induced by the toxin is proportional to the number $n(t)$ of bacteria still alive in the culture at a time t and the amount of toxin $T(t)$ remaining in the system, which is given by $\tau n(t)T(t)$, where τ is a positive constant. On the other hand, the natural growth rate of the bacteria in the culture is exponential, i.e. it grows with a rate $\gamma n(t)$, with $\gamma > 0$. In total, the number of bacteria in the culture is given by the differential equation

$$\dot{n} = \gamma n - \tau n T(t), \quad \text{for } t \geq 0.$$

(a) Find the general solution to the given linear differential equation with $n(0) = n_0$.

(b) Assume now that the toxin is injected into the system at a constant rate $T(t) = at$, where $a > 0$. Show, using a qualitative analysis of the differential equation (i.e. without solving it explicitly), that the bacterial population grows up to a time $t = \gamma/(a\tau)$, and decreases thereafter. Furthermore, show that as $t \to \infty$, $n(t) \to 0$, i.e. practically the bacterial culture is wiped out.

(c) Now find the explicit solution $n(t)$ to the differential equation and scetch $n(t)$ qualitatively as a function of t. Convince yourself that the sketch fulfils the relation between $n(t)$, $\dot{n}(t)$ and t that is specivied by the differential equation. [Check your result: if $\tau = 1, a = 1, n_0 = 1$ and $\gamma = \sqrt{\ln 2}$, then $n(\sqrt{\ln 2}) = \sqrt{2}$.]

(d) Find the time t_h at which the number of bacteria in the culture is half the initial value.

Homework Problem 3: Differential equation: substitution and separation of variables [5]

Often differential equations can be solved by convenient substitution. Here we examine differential equations of the type

$$y'(x) = f(ax + by(x) + c) . \tag{4}$$

(a) Substitute $u(x) = ax + by(x) + c$ and find a differential equation for $u(x)$.

(b) Find an implicit expression for the solution $u(x)$ of the new differential equation using an integral that contains the function f. *Hint:* Separation of variables!

(c) Use the substitution strategy of (a,b) to solve the differential equation $y'(x) = \mathrm{e}^{x+3y(x)+5}$, with initial condition $y(0) = 1$.
[Check your result: $y(\ln(e^{-8} + 3) - 2\ln 2) = \frac{1}{3}(2\ln 2 - \ln(e^{-8} + 3) - 5)$.]

(d) Check: Solve the differential equation given in (c) directly (without substitution) using separation of variables. Is the result in agreement with the result from (c)?

(e) Solve the differential equation $y'(x) = [a(x + y) + c]^2$ with initial condition $y(x_0) = y_0$ using the substitution given in (a).
[Check your result: if $x_0 = y_0 = 0$ and $a = c = 1$, then $y(0) = 0$.]

Homework Problem 4: Inhomogene linear differential equation, variation of constants [3]

Points: (a)[1](E); (b)[1](E); (c)[1](E)

The function $x(t)$ satisfies the inhomogeneous differential equation

$$\dot{x}(t) + tx(t) = \mathrm{e}^{-\frac{t^2}{2}}, \qquad \text{with initial condition} \quad x(0) = x_0. \tag{5}$$

(a) Find the solution $x_h(t)$ of the corresponding homogeneous equation with $x_h(0) = x_0$.

(b) Find the particular solution $x_p(t)$ of the inhomogeneous equation (5), with $x_p(0) = 0$ using variation of constants, $x_p(t) = c(t)x_h(t)$. What is the overall solution?
[Check your result: if $x_0 = 0$, then $x(1) = e^{-1/2}$.]

(c) For a differential equation of the form $\dot{x}(t) + a(t)x(t) = b(t)$ (ordinary, order 1, linear and inhomogeneous), the sum of the homogeneous and inhomogeneous solutions has the form:

$$x(t) = x_h(t) + x_p(t) = x_h(t) + c(t)x_h(t) = (1 + c(t))x_h(t) = \tilde{c}(t)x_h(t)\,.$$

The initial condition $x(0) = x_0$ can therefore also be satisfied by imposing on $x_h(t)$ and $\tilde{c}(t)$ the initial conditions $x_h(0) = 1$ and $\tilde{c}(0) = x_0$. Use this approach to construct a solution to the differential equation (5) of the form $x(t) = \tilde{c}(t)x_h(t)$. Does the result agree with the result as obtained in (b)? (*Learning objective of (c):* Realize that the same initial condition can be implemented in more than one way.)

Homework Problem 5: Inhomogeneous linear differential equation of order 3 [5]
Points: (a)[1](E); (b)[2](M); (c)[2](M).

Consider the following inhomogeneous linear differential equation of order 3:

Differential equation: $$\dddot{x} - 6\ddot{x} + 11\dot{x} - 6x = f_{\rm A}(t), \tag{6}$$

Initial value: $$x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = a\,, \qquad \text{with} \quad a \in \mathbb{R}. \tag{7}$$

Driving: $$f_{\rm A}(t) = \begin{cases} \mathrm{e}^{-bt} & \text{for} \quad t \geq 0\,, \\ 0 & \text{for} \quad t < 0\,, \end{cases} \qquad \text{with} \quad 0 < b \in \mathbb{R}. \tag{8}$$

For $t > 0$, find a general solution to this equation of the form $x(t) = x_{\rm h}(t) + x_{\rm p}(t)$, where $x_{\rm h}(t)$ and $x_{\rm p}(t)$ are the homogeneous and particular solutions to the homogeneous and inhomogeneous differential equation that have the initial values (7) or $x_{\rm p}(0) = \dot{x}_{\rm p}(0) = \ddot{x}_{\rm p}(0) = 0$ respectively. Proceed as follows:

(a) Write the differential equation (6) in the matrix form

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} + \mathbf{b}(t), \quad \text{with} \;\; \mathbf{x} \equiv (x, \dot{x}, \ddot{x})^T \equiv (x^1, x^2, x^3)^T, \;\; \mathbf{x}_0 = (x(0), \dot{x}(0), \ddot{x}(0))^T\,. \tag{9}$$

(b) Find the homogeneous solution $\mathbf{x}_{\rm h}(t)$ of $(9)|_{\mathbf{b}(t)=0}$ with $\mathbf{x}_{\rm h}(0) = \mathbf{x}_0$; then $x_{\rm h}(t) = x^1_{\ \rm h}(t)$.

(c) Find the inhomogeneous solution $\mathbf{x}_{\rm p}(t)$ of (9), with $\mathbf{x}_{\rm p}(0) = \mathbf{0}$; then $x_{\rm p}(t) = x^1_{\ \rm p}(t)$.

Hint: This problem is the direct analogue of the example problem on the driven, damped harmonic oscillator. The eigenvalues λ_1, λ_2, λ_3 of A are integers, with $\lambda_1 = 1$.

Homework Problem 6: System of differential equations with non-diagonizable matrix [Bonus]
Points: (a)[1](E); (b)[1](M); (c)[1](A); (d)[0.5](E); (e)[0.5](E). (alle Bonus)

We consider a procedure to solve the differential equation

$$\dot{\mathbf{x}} = A \cdot \mathbf{x} \tag{10}$$

for the case of a matrix $A \in \mathrm{Mat}(\mathbb{R}, n, n)$ that has $n-1$ distinct eigenvalues λ_j and associated eigenvectors $\mathbf{v}_j$, with $j = 1, \ldots, n-1$, where the eigenvalue λ_{n-1} is a two-fold zero of the characteristic polynomial. Since λ_{n-1} has only one eigenvector, this matrix not diagonalizable. However, it can be brought into the so-called Jordan normal form:

$$S^{-1}AS = J, \qquad J = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \lambda_{n-1} & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_{n-1} \end{pmatrix}, \qquad S = (\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n)\,. \tag{11}$$

Using $A = SJS^{-1}$, as well as $\mathbf{v}_j = S\mathbf{e}_j$ and $J\mathbf{e}_j = \lambda_j \mathbf{e}_j + \delta_{jn}\mathbf{e}_{j-1}$, one finds that this is equivalent to

$$A \cdot \mathbf{v}_j = \lambda_j \mathbf{v}_j + \mathbf{v}_{j-1}\delta_{jn}, \quad \forall j = 1, \ldots, n. \tag{12}$$

For $j = 1, \ldots, n-1$ this corresponds to the usual eigenvalue equation, and $\mathbf{v}_j$ to the usual eigenvectors. $\mathbf{v}_n$, however, is not an eigenvector, but determined by the following equation:

$$(A - \mathbb{1}\lambda_n)\mathbf{v}_n = \mathbf{v}_{n-1}. \tag{13}$$

Since $(A - \mathbb{1}\lambda_n)$ is not invertible, this equation does not uniquely fix the vector $\mathbf{v}_n$. Different choices of $\mathbf{v}_n$ lead [via (11)] to different similarity transformation matrices S, but they all yield the same form for the Jordan-Matrix J.
The λ_j and $\mathbf{v}_j$ thus obtained can be used to find a solution for the DEQ (10), using an exponential Ansatz together with 'variation of the constants':

$$\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{v}_j e^{\lambda_j t} c^j(t), \quad \text{with} \quad \lambda_n \equiv \lambda_{n-1}. \tag{14}$$

The coefficients $c^j(t)$ can be determined by inserting this Ansatz into (10):

$$0 = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} - A\right)\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{v}_j e^{\lambda_j t}\left[\lambda_j c^j(t) + \dot{c}^j(t) - \lambda_j c^j(t)\right] - \mathbf{v}_{n-1} e^{\lambda_n t} c^n(t). \tag{15}$$

Comparing coefficients of $\mathbf{v}_j$ we obtain:

$$\mathbf{v}_{j\neq n-1}: \qquad \dot{c}^j(t) = 0 \qquad \Rightarrow \qquad \boxed{c^j(t) = c^j(0) = \text{const.}}\,, \tag{16}$$

$$\mathbf{v}_{n-1}: \qquad \dot{c}^{n-1}(t) = c^n(t) \qquad \Rightarrow \qquad \boxed{c^{n-1}(t) = c^{n-1}(0) + t\, c^n(0)}\,. \tag{17}$$

The values of $c^j(0)$ are fixed by the initial condictions $\mathbf{x}(0)$:

$$\mathbf{x}(0) = \sum_j \mathbf{v}_j c^j(0) = S\mathbf{c}(0)\,, \quad \Rightarrow \quad \mathbf{c}(0) = S^{-1}\mathbf{x}(0)\,. \tag{18}$$

Now use this method to find the solution of the DEQ

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{with} \quad A = \frac{1}{3}\begin{pmatrix} 7 & 2 & 0 \\ 0 & 4 & -1 \\ 2 & 0 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{19}$$

(a) Show that the characteristic polynomial for A has a simple zero, say λ_1, and a two-fold zero, say $\lambda_2 = \lambda_3$.

(b) Show that the eigenspaces associated with λ_1 and λ_2 are both one-dimensional (which implies that A is not diagonalizable), and find the corresponding normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Use Eq. (13) to find a third, normalized vector $\mathbf{v}_3$, having the property that A is brought into a Jordan normal form using $S = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. While doing so, exploit the freedom of choice that is available for $\mathbf{v}_3$ to choose the latter orthonormal to $\mathbf{v}_1$ and $\mathbf{v}_2$. [*Remark:* For the present example orthonormality is achievable (and useful, since then $S^{-1} = S^T$ holds), but this is not generally the case.]

(d) Now use an Ansatz of the form (14) to find the solution $\mathbf{x}(t)$ to the DEQ (19). [Check your result: $\mathbf{x}(\ln 2) = (2, 4, 0)^T + \frac{4}{3}(1 + \ln 2)(2, -1, 2)^T$.]

(e) Check your result explicitly by verifying that it satisfies the DEQ.

Homework Problem 7: Integration by partial fraction expansion [4]
Points: (a)[2](M); (b)[2](M).

Use partial fraction expansions to compute the following integrals, for $z \in \mathbb{R}$, $z < 1$:

(a) $\quad I(z) = \int_0^z \mathrm{d}x \frac{x+2}{x^3 - 3x^2 - x + 3}\,,$ $\qquad$ (b) $\quad I(z) = \int_0^z \mathrm{d}x \, \frac{4x-1}{(x+2)(x-1)^2}\,.$

[Check your results: (a) $I(\frac{1}{2}) = \frac{5}{8}\ln 5 - \frac{1}{2}\ln 3$, (b) $I(\frac{1}{2}) = 1 - \ln\left(\frac{5}{2}\right)$.]

[Total Points for Homework Problems: 23]
