LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Fakultät für Physik
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Sheet 11.4: Delta Function and Fourier Series

Posted: Friday, 08.01.16 Due: Friday, 15.01.16, 13:00 Central Tutorial: 27.01.16
(b)[2](E/M/A) means: problem (b) counts 2 points and is easy/medium hard/advanced
Example problems: {T}: will be discussed in tutorial; {S}: self study.

Example Problem 1: Integrals with δ function [3]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](M); (d)[1](M). [T]

Calculate the following integrals (with $a \in \mathbb{R}$):

(a) $I_1(a) = \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-\pi) \sin(ax)$

(b) $I_2(a) = \int_{\mathbb{R}^3} \mathrm{d}^3x\, \delta(\mathbf{x}-\mathbf{y})\, \mathbf{x}^2$, with $\mathbf{y} = (a,1,2)^T$

(c) $I_3(a) = \int_0^a \mathrm{d}x\, \delta(x-\pi) \dfrac{1}{a+\cos^2(x/2)}$

(d) $I_4(a) = \int_0^3 \mathrm{d}x\, \delta(x^2-6x+8)\sqrt{\mathrm{e}^{ax}}$

[Check your results: $I_1(\frac{1}{2}) = 1$, $I_2(1) = 6$, $I_3(\pi) = \frac{1}{2\pi}$, $I_4(\ln 2) = 1$.]

Example Problem 2: Lorentz representation of the Dirac delta function [4]
Points: [4](M). [T]

Show that in the limit $\epsilon \to 0^+$, the Lorentz peak function $\delta^{[\epsilon]}(x)$ given below is a representation of the Dirac delta function $\delta(x)$. To this end, calculate (i) the height, (ii) the width x_b (defined by $\delta^{[\epsilon]}(x_\mathrm{b}) = \frac{1}{2}\delta^{[\epsilon]}(0)$, $x_\mathrm{b} > 0$) and (iii) the area of the peak. Furthermore, calculate the functions $\theta^{[\epsilon]}(x) = \int_{-\infty}^{x} \mathrm{d}x' \delta^{[\epsilon]}(x')$ and $\delta'^{[\epsilon]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\delta^{[\epsilon]}(x)$. Scetch all three functions $\theta^{[\epsilon]}$, $\delta^{[\epsilon]}$, $\delta'^{[\epsilon]}$ (above each other, using the same scaling for the x-axis).

Lorentz-Peak: $\delta^{[\epsilon]}(x) = \dfrac{\epsilon/\pi}{x^2+\epsilon^2}$.

Hint: When calculating the peak weight, use the substitution $x = \epsilon \tan y$.
Remark: Lorentzian functions are common in physics. Example: the energy of a discrete quantum state, which is weakly coupled to the environment, has the form of a Lorentzian function, the width of which is determined by the strength of the coupling to the environment. As the coupling strength approaches zero, we obtain a δ peak.

Example Problem 3: Series representation of the coth function [1]
Points: [1](E). [S]

Show that the series $\sum_{n\in\mathbb{Z}} \mathrm{e}^{-y|n|}$, with $0 < y \in \mathbb{R}$, converges to the $\coth$ function.

Example Problem 4: Fourier series of the sawtooth function [2]
Points: [2](M). [T]

Let $f(x)$ be a sawtooth function, defined by $f(x) = x$ for $-\pi < x < \pi$, $f(\pm\pi) = 0$ and $f(x+2\pi) = f(x)$. Calculate the Fourier coefficients $\tilde{f}_n$ in the representation $f(x) = \frac{1}{L}\sum_n \mathrm{e}^{\mathrm{i}k_n x}\tilde{f}_n$. How should k_n and L be chosen? Sketch the function $f(x)$, as well as the sum of the $n = 1$ and $n = -1$ terms of the Fourier series (i.e. the first term of the corresponding sine series). [Check your result: $\tilde{f}_6 = \frac{1}{3}\mathrm{i}\pi$.]

Example Problem 5: Cosine Series [4]
Points: (a)[1](E); (b)[1](E); (c)[2](M). [T]

For the function $f : I \to \mathbb{C}$, $x \mapsto f(x)$, with $I = [-L/2, L/2]$, consider the Fourier series representation $f(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx}\tilde{f}_k$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{Z}$.

(a) Show that the Fourier coefficients are given by $\tilde{f}_k = \int_{-L/2}^{L/2} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x)$.

(b) Now let f be an even function, i.e. $f(x) = f(-x)$. Show that then the Fourier coefficients are given by $\tilde{f}_k = 2\int_0^{L/2} \mathrm{d}x\, \cos(kx) f(x)$, and furthermore, that $f(x)$ can be represented by a cosine series of the form $f(x) = \frac{1}{2}a_0 + \sum_{k>0} a_k \cos(kx)$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{N}_0$. Find a_k, expressed via $\tilde{f}_k$.

(c) Now consider the following function: $f(x) = 1$ for $|x| < L/4$, $f(x) = -1$ for $L/4 < |x| < L/2$. Scetch it, and compute the coefficients $\tilde{f}_k$ and a_k of the corresponding Fourier and cosine series.

Example Problem 6: Parseval's identity and convolution [7]
Points: (a)[3](M); (b)[2](M); (c)[2](M). [T]

Let $f(x)$ be a sawtooth function, defined by $f(x) = x$ for $-\pi < x < \pi$, $f(\pm\pi) = 0$ and $f(x + 2\pi) = f(x)$. In the Fourier representation $f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx}\tilde{f}_n$ its Fourier coefficients are $\tilde{f}_n = 2\pi\mathrm{i}(-1)^n/n$ [see example problem 4]. Let $g(x) = \sin x$.

(a) Using this concrete example, check that Parseval's identity holds, by computing both the integral $\int_{-\pi}^{\pi} \mathrm{d}x \overline{f}(x) g(x)$ and the sum $(1/2\pi)\sum_n \overline{\tilde{f}}_n\, \tilde{g}_n$ explicitly.

(b) Prove the famous identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, by on the one hand computing the integral $\int_{-\pi}^{\pi} \mathrm{d}x f^2(x)$ directly and on the other hand expressing it as a sum over Fourier modes using Parseval's identity.

(c) Calculate the convolution $(f * g)(x)$ both by directly computing the convolution integral and by using the convolution theorem and a summation of Fourier coefficients.

[Total Points for Example Problems: 21]

Homework Problem 1: Integrals with δ function [4]
Points: (a)[0.5](E); (b)[0.5](E); (c)[1](M); (d)[1](M); (e)[1](A).

Calculate the following integrals (with $a \in \mathbb{R}$, $n \in \mathbb{N}$):

(a) $I_1(a) = \displaystyle\int_1^4 \mathrm{d}x\, \delta(x-2)\, (a^x + 3)$

(b) $I_2(a) = \displaystyle\int_{\mathbb{R}^2} \mathrm{d}^2x\, \delta(\mathbf{x} - \mathbf{y})\, (x_1 + x_2)^2\, \mathrm{e}^{3-x_1}\,, \quad \text{with} \quad \mathbf{y} = (3, a)^T$

(c) $I_3(a) = \displaystyle\int_{-1}^1 \mathrm{d}x\, \sqrt{2+2x}\ \delta(ax-2)\,, \quad \text{with} \quad a \neq 0$

(d) $I_4(a) = \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(3^{-x} - 9)(1 - x^a)$

(e) $I_5(n) = \displaystyle\int_{-\pi/2}^{9\pi/2} \mathrm{d}x\ \cos(nx)\ \delta\,(\sin x)$

[Check your results: $I_1(3) = 12$, $I_2(-5) = 4$, $I_3(2) = \frac{1}{2}$, $I_4(3) = \frac{1}{\ln 3}$, $I_5(7) = 1$.]

Homework Problem 2: Representations of the Dirac delta Funktion [4]
Points: [4](M).

Show that in the limit $\epsilon \to 0^+$, the peak-shaped function $\delta^{[\epsilon]}(x)$ given below is a representation of the Dirac delta function $\delta(x)$. To this end, calculate (i) the height, (ii) the width x_b (defined by $\delta^{[\epsilon]}(x_\mathrm{b}) = \frac{1}{2}\delta^{[\epsilon]}(0)$, $x_\mathrm{b} > 0$) and (iii) the area of the peak. Furthermore, calculate the functions $\theta^{[\epsilon]}(x) = \int_{-\infty}^x \mathrm{d}x' \delta^{[\epsilon]}(x')$ and $\delta'^{[\epsilon]}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\delta^{[\epsilon]}(x)$. Sketch all three functions $\theta^{[\epsilon]}$, $\delta^{[\epsilon]}$, $\delta'^{[\epsilon]}$ (above each other, using the same scaling for the x-axis).

Gaussian peak: $\quad \delta^{[\epsilon]}(x) = \dfrac{1}{\epsilon\sqrt{\pi}} \mathrm{e}^{-(x/\epsilon)^2}\,.$

Hint: The function $\theta^{[\epsilon]}(x)$ cannot be calculated in terms of elementary functions; instead write it in terms of the 'error function', $\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}y\, \mathrm{e}^{-y^2}$, with $\mathrm{Erf}(\infty) = 1$.
Remark: Gaussians appear very often in physics. Example: A quantum mechanical harmonic oscillator with spring constant k and potential energy $\frac{1}{2}kx^2$ has a Gaussian wavefunction for its ground state, with width $\sim 1/\sqrt{k}$.

Homework Problem 3: Series representation of the periodic δ function [5]
Points: (a)[0.5](E); (b)[0.5](M); (c)[1.5](A); (d)[0.5](E); (e)[1](A); (f)[0.5](E); (g)[0.5](E)

Show that the function $\delta^{[\epsilon]}(x)$, defined by

$$\delta^{[\epsilon]}(x) = \frac{1}{L} \sum_k \mathrm{e}^{\mathrm{i}kx - \epsilon|k|}\,, \quad k = 2\pi n/L, \quad n \in \mathbb{Z}\,, \quad x, \epsilon, L \in \mathbb{R}\,, \quad 0 < \epsilon \ll L\,, \tag{1}$$

has the following properties:

(a) $$\delta^{[\epsilon]}(x) = \delta^{[\epsilon]}(x + L)\,. \tag{2}$$

(b) $$\int_{-L/2}^{L/2} \mathrm{d}x\, \delta^{[\epsilon]}(x) = 1\,. \quad \textit{Hint: } \text{Treat } k = 0 \text{ and } k \neq 0 \text{ separately in } \sum_k. \tag{3}$$

(c) $$\delta^{[\epsilon]}(x) = \frac{1}{2L}\left[\frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}}\right] = \frac{1}{L}\frac{1 - \mathrm{e}^{-4\pi\epsilon/L}}{1 + \mathrm{e}^{-4\pi\epsilon/L} - 2\mathrm{e}^{-2\pi\epsilon/L}\cos(2\pi x/L)}\,, \tag{4}$$

where $w = \mathrm{e}^{2\pi(\mathrm{i}x-\epsilon)/L}$ and $\overline{w} = \mathrm{e}^{2\pi(-\mathrm{i}x-\epsilon)/L}$.
Hint: Write out the sum in Eq. (1) as a geometric series in powers of w and $\overline{w}$.

(d) $\lim_{\epsilon\to 0} \delta^{[\epsilon]}(x) = 0$ for $x \neq mL$, with $m \in \mathbb{Z}$. *Hint:* Start from Eq. (4). (5)

(e) $\delta^{[\epsilon]}(x) \simeq \dfrac{\epsilon/\pi}{\epsilon^2 + x^2}$ for $|x|/L \ll 1$ and $\epsilon/L \ll 1$. (6)

Hint: Taylor expand the numerator in Eq. (4) up to first order in $\tilde{\epsilon} = 2\pi\epsilon/L$, and the denominator up to second order in $\tilde{\epsilon}$ and $\tilde{x} = 2\pi x/L$.

(f) Sketch the function $\delta^{[\epsilon]}(x)$ qualitatively for $\epsilon/L \ll 1$ and $x \in [-\frac{7}{2}L, \frac{7}{2}L]$.

(g) Deduce that in the limit of $\epsilon \to 0$, $\delta^{[\epsilon]}(x)$ represents a periodic δ function, with

$$\delta^{[0]}(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx} = \sum_{m\in\mathbb{Z}} \delta(x - mL) \ . \tag{7}$$

Homework Problem 4: Fourier series [4]
Points: (a)[2](E); (b)[2](M)

Determine the Fourier series for the following periodic functions, i.e. calculate the Fourier coefficients $\tilde{f}_n$ in the representation $f(x) = \frac{1}{L}\sum_n \mathrm{e}^{\mathrm{i}k_n x}\tilde{f}_n$. How should k_n and L be chosen in each case? Sketch the functions first.

(a) $f(x) = |\sin x|$, (b) $f(x) = \begin{cases} 4x & \text{for} \quad -\pi \le x < 0\,, \\ 2x & \text{for} \quad 0 \le x < \pi\,, \end{cases}$ and $f(x + 2\pi) = f(x)$.

[Check your results: (a) $\tilde{f}_3 = -\frac{2}{35}$, (b) $\tilde{f}_3 = \frac{2}{9}(2 - 9\mathrm{i}\pi)$.]

Homework Problem 5: Sine Series [3]
Points: (a)[1](E); (b)[2](M)

For the function $f : I \to \mathbb{C}$, $x \mapsto f(x)$, with $I = [-L/2, L/2]$, consider the Fourier series representation $f(x) = \frac{1}{L}\sum_k \mathrm{e}^{\mathrm{i}kx}\tilde{f}_k$, with $k = \frac{2\pi n}{L}$ and $n \in \mathbb{Z}$, with Fourier coefficients $\tilde{f}_k = \int_{-L/2}^{L/2} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} f(x)$.

(a) Let f be an odd function, i.e. $f(x) = -f(-x)$. Show that then the Fourier coefficients are given by $\tilde{f}_k = -\mathrm{i}2\int_0^{L/2} \mathrm{d}x\, \sin(kx) f(x)$, and furthermore, that $f(x)$ can be represented by a sine series of the form $f(x) = \sum_{k>0} b_k \sin(kx)$ with $k = \frac{2\pi n}{L}$ und $n \in \mathbb{N}_0$. What does b_k look like when expressed via $\tilde{f}_k$?

(b) Now consider the following function: $f(x) = 1$ for $0 < x < L/2$, $f(x) = -1$ for $-L/2 < x < 0$. Scetch it, and compute the coefficients $\tilde{f}_k$ and b_k of the corresponding Fourier and sine series.

Homework Problem 6: Convolution theorem [3]
Points: (a)[0.5](E); (b)[1](M); (c)[1.5](A)

Learning objective: This problem illustrates how a complicated sum may be calculated explicitly using the convolution theorem.
Consider the function $f_\gamma(t) = f_\gamma(0)\mathrm{e}^{\gamma t}$ for $t \in [0, \tau)$ and $f(t+\tau) = f(t)$ with $f_\gamma(0) = 1/(\mathrm{e}^{\gamma\tau} - 1)$.

(a) Consider a Fourier series representation of $f_\gamma(t)$ of the following form:

$$f_\gamma(t) = \frac{1}{\tau}\sum_{\omega_n} \mathrm{e}^{-\mathrm{i}\omega_n t}\tilde{f}_{\gamma,n}, \qquad \tilde{f}_{\gamma,n} = \int_0^\tau dt \mathrm{e}^{\mathrm{i}\omega_n t} f_\gamma(t), \quad \text{mit} \quad \omega_n = 2\pi n/\tau, \quad n \in \mathbb{Z}. \quad (8)$$

Show that the Fourier coefficients are given by $\tilde{f}_{\gamma,n} = 1/(\mathrm{i}\omega_n + \gamma)$.

(b) Use this result and the convolution theorem to express the following series as a convolution of f_γ and $f_{-\gamma}$:

$$S(t) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega_n t}}{\omega_n^2 + \gamma^2} = -\tau \int_0^\tau dt' f_\gamma\left(t - t'\right) f_{-\gamma}\left(t'\right) . \quad (9)$$

(c) Scetch the functions $f_\gamma(t - t')$ and $f_{-\gamma}(t')$ occuring in the convolution theorem as functions of t', for $t' \in [-\tau, 2\tau]$. Assume $0 \leq t \leq \tau$ and show that the convolution integral (9) is given by the following expression:

$$S(t) = \frac{\tau\left[\sinh\left(\gamma\left(t - \tau\right)\right) - \sinh\left(\gamma t\right)\right]}{2\gamma\left[1 - \cosh\left(\gamma\tau\right)\right]} .$$

Hint: The integral $\int_0^\tau \mathrm{d}t'$ involves an interval of t' values for which $t - t'$ lies outside of $[0, \tau)$. It is therefore advisable to split the integral into two parts, with $\int_0^t \mathrm{d}t'$ and $\int_t^\tau \mathrm{d}t'$.

[Total Points for Homework Problems: 23]