Spinors in 1+3 dimensions

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1 Lie groups and Lie algebras

A *Lie group* \( G \) is a group which is also a differentiable manifold where the group operations of multiplication and inversion are smooth maps. To every Lie group \( G \) we can associate a Lie algebra \( g \) whose elements lie in the tangent space at the identity element of the Lie group \( G \). Because of the smooth structure of \( G \), there exists a mapping from \( g \) to \( G \), namely the *exponential map*

\[
g = \exp[\alpha^i T_i],
\]

where \( g \in G \), \( T_i \in g \) and the \( \alpha^i \)'s are \( \mathbb{R} \)-valued parameters. However, note that the map does not have to be either injective or surjective. The \( T_i \)'s can be chosen to form a basis of the algebra \( g \) and are called *generators* of \( G \).

1.1 Homomorphisms

A *(group) homomorphism* is a map \( \Phi \) from a group \( G \) to another group \( G' \) which preserves the group operation, i.e., for any \( g, h \in G \) one has \( \Phi(gh) = \Phi(g)\Phi(h) \). Moreover, a bijective homomorphism is called *isomorphism*. A particular type of homomorphisms are the *(linear) representations* \( R_n \{ G \} \) of \( G \), that is, the homomorphisms mapping \( G \) to a subgroup of the *general linear group* \( GL(n, \mathbb{K}) \), the set of all invertible \( n \times n \) matrices on a field \( \mathbb{K} = \mathbb{R}, \mathbb{C} \). Then, the particular representation acts on an \( n \)-dimensional vector space \( V_n \) over \( \mathbb{K} \). Note that the commutation relations of the generators \( T_i \) of \( G \):

\[
[T_i, T_j] = if_{ijk} T_k,
\]

where \( f_{ijk} \) are called *structure constants*, are independent on the chosen representation and define therefore completely the Lie algebra \( g \) associated to \( G \). By the exponential map (1), one could obtain representations of the Lie group \( G \) by finding representations of the associated Lie algebra \( g \).

Take a representation \( R_n \{ G \} \) which acts on an \( n \)-dimensional vector space \( V_n \). If, by acting on any element
of the subspace $W_m$ with any group element in the representation $\mathcal{R}_n\{G\}$, the resulting element is still in $W_m$, then the subspace $W_m$ is called an invariant subspace of $V_n$. If the only invariant subspaces of $V_n$ are given by the zero vector space $\{0\}$ and the vector space itself $V_n$, then the representation $\mathcal{R}_n\{G\}$ is called an irreducible representation.

A completely reducible representation is a representation which can be written as a direct sum of irreducible representations, i.e., for any $g \in G$, $\mathcal{R}_n(g)$ can be written in block diagonal form where any block relates to an irreducible representation. Note however that, if a representation is reducible, it does not mean that it is also completely reducible. Unitary representations (where $\mathcal{R}_n^\dagger(g) = \mathcal{R}_n^{-1}(g)$ for all $g \in G$) are always completely reducible.

On the other hand, for any reducible representation $\mathcal{R}_n\{G\}$, $\mathcal{R}_n(g)$ can at least be written as an upper triangular block matrix.

## 2 The Lorentz group

An example of a Lie group is the Lorentz group $O(1,3)$, the manifold of $\mathbb{R}$-valued $4 \times 4$ matrices $\Lambda^\mu_\nu$ which are orthogonal, that is, they leave the Minkowski metric $\eta = \text{diag}(+, -, -, -)$ invariant:

$$\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu = \Lambda^T \eta \Lambda = (\Lambda^T \eta \Lambda)_{\mu\nu}. \tag{3}$$

The proper, orthochronous Lorentz group in 1+3 dimensions $SO^+(1,3)$ is the connected submanifold of $O(1,3)$ containing the identity transformation. It can be characterized as the set of all the orthogonal $\mathbb{R}$-valued $4 \times 4$ matrices with unit determinant and with $\Lambda^0_0 \geq 1$. An infinitesimal Lorentz transformation acts as

$$\Lambda^\mu_\nu x^\nu = x^\mu + \omega^\mu_\nu x^\nu, \tag{4}$$

where $\omega^\rho_\sigma$ is an $\mathbb{R}$-valued tensor. By means of (3), one gets

$$\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu \simeq \eta_{\rho\sigma} (\delta^\rho_\mu + \omega^\rho_\mu) (\delta^\sigma_\nu + \omega^\sigma_\nu) = \eta_{\mu\nu} + \omega^\mu_\nu + \omega^\nu_\mu \equiv \eta_{\mu\nu}. \tag{5}$$

Therefore, $\omega^\rho_\sigma$ has to be antisymmetric. The Lie algebra $\mathfrak{so}(1,3)$ of $SO^+(1,3)$ is spanned by the six generators $J_i, K_i \in \mathfrak{so}(1,3), \ i = 1, 2, 3$, associated, respectively, to spatial rotations and boosts. The commutation relations of the generators are the following:

$$[J_i, J_j] = i \epsilon_{ijk} J_k, \quad [J_i, K_j] = i \epsilon_{ijk} K_k, \quad [K_i, K_j] = -i \epsilon_{ijk} J_k. \tag{6}$$

One can redefine the 6 generators $J_i, K_i$ by making use of the fact that a $4 \times 4$ antisymmetric matrix $L_{\mu\nu} = -L_{\nu\mu}$ has exactly 6 independent entries. By setting $L_{ij} = \epsilon_{ijk} J_k$ and $L_{i0} = K_i$, the commutation relations (6) become

$$[L_{\mu\nu}, L_{\rho\sigma}] = i \eta_{\rho\sigma} L_{\mu\nu} + i \eta_{\nu\rho} L_{\mu\sigma} - i \eta_{\mu\rho} L_{\nu\sigma} - i \eta_{\mu\sigma} L_{\nu\rho}. \tag{7}$$

In complete analogy to (1), the exponential map in the defining representation can be written as

$$\Lambda^\mu_\nu = \exp\left[\frac{i}{2} \omega^\rho_\sigma (L_{\rho\sigma})^\mu_\nu\right]. \tag{8}$$

Note that, owing to the reality of $\Lambda^\mu_\nu$, the generators in the defining representation $(L_{\rho\sigma})^\mu_\nu$ have to be imaginary. Putting together (4) and (8) and expanding up to first order in $\omega^\rho_\sigma$, yields

$$x^\mu + \omega^\rho_\sigma x^\sigma = x^\mu + \frac{1}{2} (\delta^\mu_\rho \eta_{\sigma\nu} - \delta^\mu_\sigma \eta_{\rho\nu}) \omega^\rho_\sigma x^\nu \equiv x^\mu + \frac{1}{2} \omega^\rho_\sigma (L_{\rho\sigma})^\mu_\nu. \tag{9}$$

Therefore, we can identify

$$(L_{\rho\sigma})^\mu_\nu = -i (\delta^\rho_\mu \eta_{\sigma\nu} - \delta^\rho_\sigma \eta_{\mu\nu}), \tag{10}$$

which are the generators of the Lorentz group in the defining representation. Indeed, (10) satisfies the Lorentz algebra given in (7).

## 3 The special linear group

The Lorentz group $SO^+(1,3)$ has the special linear group $SL(2, \mathbb{C})$ as its double cover, i.e., there exists a double valued surjective homomorphism $\Phi : SL(2, \mathbb{C}) \rightarrow SO^+(1,3)$. For any 4-vector $x^\mu$ and $M \in SL(2, \mathbb{C})$, define

$$y^\mu \sigma_\mu := M x^\nu \sigma_\nu M^\dagger, \tag{11}$$
where $M^\dagger$ is the adjoint matrix of $M$ and

$$
\sigma_{\mu} = (\mu, \sigma) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right)
$$

are the generalized Pauli matrices. It is then possible to show that $y^\mu$ corresponds to a Lorentz transformed 4-vector and that every element of $SO^+(1,3)$ is related to two elements of $SL(2, \mathbb{C})$, namely $M$ and $-M$, by a homomorphism. Due to the covering, the two groups are locally (in a neighborhood of the identity element) isomorphic, that is, $so(1, 3) \cong sl(2, \mathbb{C})$. Therefore, the generators of $SL(2, \mathbb{C})$ also satisfy the commutation relations (6) and (7). Indeed, taking into account (1), one gets

$$
1 = \det M = \det \exp[i(\alpha^i J_i + \beta^j K_j)] = \exp[i \text{ tr}(\alpha^i J_i + \beta^j K_j)].
$$

Since the $\mathbb{R}$-valued parameters $\alpha^i, \beta^j$ are arbitrary, the generators of $SL(2, \mathbb{C})$ have to be traceless. A basis for the space of traceless complex $2 \times 2$ matrices is given by the hermitian Pauli matrices $\sigma_i$ and the antihermitian Pauli matrices $i \sigma_i$. By setting

$$
J_i = \frac{\sigma_i}{2} \quad \text{and} \quad K_i = \pm i \frac{\sigma_i}{2},
$$

those clearly satisfy the commutation relations (6). The circumstance that we can not choose all generators to be hermitian derives from the fact that $SL(2, \mathbb{C})$, as well as the Lorentz group, is not topologically compact. Let us introduce a change of basis defined by

$$
J_i^\pm := \frac{1}{2} (J_i \pm i K_i).
$$

Then, the commutation relations (6) become

$$
[J_i^\pm, J_j^\pm] = i \epsilon_{ij} k J_k^\pm, \quad [J_i^\pm, J_j^\mp] = 0,
$$

which, as we will see in section 4.1, resemble two disjoint sets of generators of $SU(2)$.

### 3.1 The fundamental representation

The fundamental representation of $SL(2, \mathbb{C})$ is given by $2 \times 2$ matrices with unit determinant. Those matrices act naturally on a $\mathbb{C}$-valued 2-dimensional object $\psi_\alpha$, $\alpha = 1, 2$ which we call a (left-handed Weyl) spinor. Take $M \in SL(2, \mathbb{C})$, then the transformation rule is defined as

$$
\psi_\alpha \rightarrow \psi'_\alpha = M^{\beta}_{\alpha} \psi_\beta.
$$

Throughout this survey, consider that the positions of the spinor indices are of vital importance. Moreover, bear in mind that the "natural way" to contract spinor indices is from upper left to lower right. The reason will become clear in section 3.3.

### 3.2 The dual representation

This section aims to introduce the dual representation and to show that the fundamental representation is equivalent to the former, i.e., there exists a similarity transformation which maps $M$ to $M^{-1 \dagger}$. Indeed, take $M, N, O \in SL(2, \mathbb{C})$ such that $MN = O$, then

$$
M^{-1 \dagger} N^{-1 \dagger} = (N^{-1} M^{-1})^T = (MN)^{-1 \dagger} = O^{-1 \dagger}.
$$

This representation acts on another kind of spinor, the dual (Weyl) spinor $\psi^\alpha$, which transforms under an $M \in SL(2, \mathbb{C})$ as

$$
\psi^\alpha \rightarrow \psi'^\alpha = (M^{\dagger \alpha}_{\beta})^\alpha \psi^\beta = \psi^\beta (M^{-1})^\alpha_{\beta}.
$$

It is now possible to define an inner product

$$
\langle \phi, \psi \rangle \equiv \phi^\alpha \psi^\alpha := \phi^\alpha \psi_\alpha,
$$

which is $SL(2, \mathbb{C})$-invariant:

$$
\phi^\alpha \psi_\alpha \rightarrow \phi'^\alpha \psi'_\alpha = \phi^\beta (M^{-1})^\alpha_{\beta} M^\gamma_{\alpha} \psi_\gamma = \phi^\alpha \psi_\alpha,
$$

where $M^\dagger$ is the adjoint matrix of $M$ and
Let us now introduce the \( \epsilon \) tensor. As we have seen, we can use the epsilon tensor to lower spinor indices. It would be useful to find an object \( M \) that \( \delta^\gamma_\delta \epsilon^{\alpha\beta} = \delta^\gamma_\delta \epsilon^{\alpha\beta} \) as one can see from
\[
\delta^\alpha_\beta \rightarrow \delta^\prime_\alpha_\beta = M^\gamma_\delta \delta^\gamma_\delta (M^{-1})^\beta_\delta = M^\gamma_\delta (M^{-1})^\gamma_\delta = \delta^\alpha_\beta. \tag{22}
\]
Let us now introduce the \emph{epsilon tensor}
\[
\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{12} := 1. \tag{23}
\]
Furthermore, denote the inverse of the epsilon tensor with \( \epsilon^{\alpha\beta} \), fixed by the relation
\[
\epsilon^{\alpha\gamma} \epsilon_{\gamma\alpha} = \delta^\alpha_\alpha \quad \text{with} \quad \delta^\alpha_\beta \epsilon_{\alpha\beta} = \psi_\alpha. \tag{24}
\]
However, as we will see in section 3.3, \( \delta^\alpha_\beta = -\delta^\beta_\alpha \). By inspection, one can check that such an object exists and is given by
\[
\epsilon^{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{12} = 1. \tag{25}
\]
For \( M \in SL(2, \mathbb{C}) \), \( \epsilon_{\alpha\beta} \) transforms as
\[
\epsilon_{\alpha\beta} \rightarrow \epsilon'_{\alpha\beta} = M^\gamma_{\alpha} M^\delta_{\beta} \epsilon_{\gamma\delta}. \tag{26}
\]
It is easy to see that also \( \epsilon'_{\alpha\beta} \) is antisymmetric and has therefore to be a multiple of the epsilon tensor, i.e.
\[
\epsilon'_{\alpha\beta} = \eta \epsilon_{\alpha\beta}. \quad \text{By evaluating the latter formula for } \alpha = 1, \beta = 2, \text{ one gets}
\eta = \eta \epsilon_{12} = M^1_1 M^2_2 \epsilon_{12} + M^1_2 M^2_1 \epsilon_{21} = M^1_1 M^2_2 - M^1_2 M^2_1 = \det[M] = 1. \tag{27}
\]
We conclude that the epsilon tensor is an invariant tensor under \( SL(2, \mathbb{C}) \) transformations (a similar analysis applied to \( \epsilon^{\alpha\beta} \) yields also invariance under \( SL(2, \mathbb{C}) \) of the latter). Consider, for any \( N \in GL(2, \mathbb{C}) \), the identity
\[
\det(N) \epsilon_{\alpha\beta} = N^\gamma_\alpha N^\delta_\beta \epsilon_{\gamma\delta}. \tag{28}
\]
Therefore, we can equivalently define \( SL(2, \mathbb{C}) \) as the set of all \( 2 \times 2 \) matrices which leave the epsilon tensor invariant. Now assume the existence of a non-antisymmetric \( SL(2, \mathbb{C}) \)-invariant tensor \( \rho_{\alpha\beta} \), that is,
\[
\rho_{\alpha\beta} = M^\gamma_{\alpha} M^\delta_{\beta} \rho_{\gamma\delta} \quad \text{with} \quad \rho_{\alpha\beta} \neq -\rho_{\beta\alpha} \tag{29}
\]
for any \( M \in SL(2, \mathbb{C}) \). This would imply another constraint on the matrices \( M \) generating a true subgroup of \( SL(2, \mathbb{C}) \). We conclude that, up to a scalar prefactor, the \( \epsilon \)'s are the only non-trivially invariant tensors of rank two of the double cover of the Lorentz group. One can restate the invariance of the epsilon tensor as
\[
(M^{-1T})^\gamma_\beta = -\epsilon^{\alpha\gamma} M^\delta_{\gamma} \epsilon_{\delta\beta}, \tag{30}
\]
or, shorthand,
\[
M^{-1T} = \epsilon^{-1} M? \epsilon. \tag{31}
\]
This is the aforementioned similarity transformation which describes an isomorphism relating the fundamental with the dual representation.

### 3.3 Handling spinor indices

Let us examine how the contraction \( \psi^\beta \epsilon_{\beta\alpha} \) transforms under \( M \in SL(2, \mathbb{C}) \):
\[
\psi^\beta \epsilon_{\beta\alpha} \rightarrow \psi'^\beta \epsilon'_{\beta\alpha} = (M^{-1T})^\beta_\alpha \psi^\gamma M^\gamma_\beta \epsilon_{\gamma\delta} = \psi^\gamma M^\gamma_\alpha \epsilon_{\gamma\delta} = \psi^\gamma \epsilon_{\gamma\delta}, \tag{32}
\]
i.e., the contraction transforms as a spinor. Due to the invariance of the epsilon tensor, note that \( \psi^\beta \epsilon_{\beta\alpha} = \psi^\beta \epsilon_{\beta\alpha} \). That is, the transformed spinor corresponds to the contraction with the beforehand transformed dual spinor. Therefore, we can identify
\[
\psi_{\alpha} \equiv \psi^\beta \epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta} \psi^\beta. \tag{33}
\]
As we have seen, we can use the epsilon tensor to lower spinor indices. It would be useful to find an object \( \epsilon^{\alpha\beta} \) capable of raising the spinor indices:
\[
\psi^\alpha := \epsilon^{\alpha\beta} \psi_\beta. \tag{34}
\]
First, note that
\[ \epsilon_{\alpha\beta} = \epsilon^\delta_{\gamma\alpha} \epsilon^\epsilon_{\delta\beta} = \delta^\delta_{\alpha\beta} = \epsilon_{\alpha\beta}, \]  
\tag{35}
i.e., \( \epsilon^{\alpha\beta} \) can be recovered by raising the indices of \( \epsilon_{\alpha\beta} \) and vice versa. For consistency, let us verify if lowering and subsequently raising a spinor index corresponds to the identity operator. Indeed, by using equation (24) one gets
\[ \psi_\alpha = \psi^\gamma \epsilon_{\gamma\alpha} = \epsilon^\gamma_{\beta} \psi_\beta = \delta_\alpha^\beta \psi_\beta = \psi_\alpha. \]  
\tag{36}
One can easily check that also the converse is true, i.e., raising and subsequently lowering leads to the same dual spinor. Observe that if we assume \( \delta^\beta_{\alpha} = \delta^\alpha_{\beta} \) then one has
\[ \psi_\alpha = \psi^\gamma \epsilon_{\gamma\alpha} = \epsilon^\gamma_{\beta} \psi_\beta = -\epsilon_{\alpha\gamma} \epsilon^\gamma_{\beta} \psi_\beta = -\delta^\beta_{\alpha} \psi_\beta = -\psi_\alpha. \]  
\tag{37}
This is a pretty awkward contradiction. However, a way to bypass this issue is easily found. So far we contracted only from upper left to lower right, the introduction of a contraction from lower left to upper right requires that a minus sign pops out in the calculation. This gives for example
\[ \delta^\beta_{\alpha} = \epsilon^\delta_{\alpha\gamma} \epsilon_{\gamma\beta} = -\delta^\beta_{\alpha} \]  
\tag{38}
and
\[ \phi^\alpha \psi_\alpha = -\phi_\alpha \psi^\alpha. \]  
\tag{39}

### 3.4 The antifundamental representation and its dual

The fundamental and its dual representation are not the only (up to similarity transformations) 2-dimensional representations of \( SL(2, \mathbb{C}) \). For \( M \in SL(2, \mathbb{C}) \), define the complex conjugation as
\[ \overline{M}^{\beta}_{\alpha} := (M^\alpha_{\beta})^*. \]  
\tag{40}
but with the convention that the “natural way” of contracting dotted spinor indices is, contrary to the undotted ones, from lower left to upper right. This gives the antifundamental representation of the double cover of the Lorentz group. However, be aware that the definition (40) might be different in some textbooks. The dots on the indices are introduced merely as a mnemonic device to distinguish the fundamental from the antifundamental representation, in order to avoid, for example, contractions between dotted and undotted indices. Analogously to the above, \( M^{-1T} \)defines the dual of the antifundamental representation. According to the definitions given in (17) and (19), those representations act on (right-handed Weyl) spinors (with bars on top) in the following way:
\[ \overline{\psi}_\alpha \rightarrow \overline{\psi}^\alpha = \overline{\psi}_\beta (M^{\beta}_{^\gamma\alpha}) \quad \text{and} \quad \overline{\psi}^\alpha \rightarrow \overline{\psi}^{\beta\alpha} = (M^{-1})^\alpha_{\beta} \overline{\psi}_\beta. \]  
\tag{41}
Therefore, we can identify
\[ \overline{\psi}_\alpha = (\psi_\alpha)^*. \]  
\tag{42}
The complex conjugate of the epsilon tensor is given by
\[ (\epsilon_{\alpha\beta})^* = \epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}. \]  
\tag{43}
and equivalently for its inverse. We see that, bearing in mind the adopted conventions, all the discussions made for the fundamental representation and its dual apply also to the antifundamental representation and its dual. In particular, remark that equations (33), (34) and (38) become, respectively,
\[ \overline{\psi}_\alpha = -\epsilon_{\alpha\beta} \overline{\psi}_\beta, \quad \overline{\psi}^{\alpha\beta} = -\epsilon^{\alpha\beta} \epsilon_{\gamma\beta} \overline{\psi}^\gamma, \quad \overline{\psi}^{\alpha\beta} = \overline{\psi}_\beta \delta^\beta_{\alpha} = -\delta^\beta_{\alpha} \overline{\psi}_\beta = \overline{\psi}_\alpha. \]  
\tag{44}
Additionally, by using (30) one gets
\[ (M^{-1T})^\alpha_{\beta} = [(M^{-1})^\alpha_{\beta}]^* = -\epsilon_{\alpha\gamma} M^\gamma_{\delta\beta} \epsilon_{\delta\beta} = -\epsilon_{\alpha\gamma} M^\gamma_{\delta\beta} \epsilon_{\delta\beta} = (\epsilon^{-1} M)_{\alpha\beta}. \]  
\tag{45}
Therefore, the latter two representations are equivalent to one another, but not equivalent to the former two since there does not exist an invariant tensor which relates the representations by a similarity transformation. When considering fundamental representations of \( SL(2, \mathbb{C}) \), one has to deal with two inequivalent representations.
3.5 Grassmann numbers

Let us commute the spinors in the inner product introduced in (20):

$$\phi \psi = \phi^\alpha \psi_\alpha = \epsilon^{\alpha\beta} \phi_\beta \psi_\alpha = -\epsilon^{\alpha\beta} \psi_\alpha \phi_\beta = -\psi^\alpha \phi_\alpha = -\psi \phi$$  \hspace{1cm} (46)

and equivalently $\bar{\psi} \bar{\phi} = \bar{\phi}_\alpha \bar{\psi}^\alpha = -\bar{\psi} \phi$. Everything so far has been classical and therefore we treated the entries of the spinors as $c$-numbers ($c$ stands for “commuting”). When going to the quantum mechanical regime, one promotes the spinors to fermionic operators, i.e., operators satisfying the anticommutation relations

$$\{\psi_\alpha, \phi_\beta\} = \{\bar{\psi}_\alpha, \bar{\phi}_\beta\} = \{\bar{\psi}_\alpha, \phi_\beta\} = 0 \hspace{1cm} (47)$$

but which are still commuting with $c$-numbers. In particular, it follows that $\psi_1 \psi_1 = \psi_2 \psi_2 = 0$ and etc.. The anticommutation relations in (47) define a Grassmann algebra and the entries of the spinors are therefore called Grassmann numbers. Equation (46) now becomes

$$\phi \psi = \phi^\alpha \psi_\alpha = \epsilon^{\alpha\beta} \phi_\beta \psi_\alpha = \epsilon^{\alpha\beta} \psi_\alpha \phi_\beta = \psi^\alpha \phi_\alpha = \psi \phi, \quad \bar{\phi} \bar{\psi} = \bar{\psi} \phi. \hspace{1cm} (48)$$

Under complex conjugation we set the rule that the Grassmann numbers interchange:

$$(\phi_\alpha \psi_\beta)^* = (\psi_\beta)^* (\phi_\alpha)^* = \bar{\psi}^\beta \bar{\phi}_\alpha \hspace{1cm} (49)$$

such that

$$(\phi \psi)^* = (\phi^\alpha \psi_\alpha)^* = (\psi^\alpha)^* (\phi_\alpha)^* = \bar{\psi}_\alpha \bar{\phi}^\alpha = \bar{\psi} \bar{\phi} = \bar{\phi} \bar{\psi}. \hspace{1cm} (50)$$

3.6 Tensorial representations

A tensorial representation of rank $s + t$ of $SL(2, \mathbb{C})$ is given by a tensor $Q_{\alpha_1...\alpha_s,\dot{\beta}_1...\dot{\beta}_t}$. As we will see later, tensorial representations allow for formulating representations for any spin $s \in \frac{1}{2} \mathbb{Z}$. However, general tensorial representations are reducible. To make them irreducible, one has to impose a bunch of relations on the tensor $Q$. A didactical example of tensorial representation is given by the bispinor $Q_{\alpha\dot{\alpha}}$, a tensor of rank $1 + 1$. It transforms according to

$$Q_{\alpha\dot{\alpha}} \rightarrow Q'_{\alpha\dot{\alpha}} = M_{\alpha}^{\beta} Q_{\beta\dot{\alpha}} (M^T)^{\dot{\beta}}_{\dot{\alpha}} = M_{\alpha}^{\beta} Q_{\beta\dot{\alpha}} (M^1)^{\dot{\beta}}_{\dot{\alpha}}, \hspace{1cm} (51)$$

that is,

$$Q \rightarrow Q' = MQM^\dagger. \hspace{1cm} (52)$$

By means of equation (11), this corresponds to a Lorentz transformation. Therefore, the bispinor is nothing other than $Q = x^\mu \sigma_\mu$ for a specific 4-vector $x^\mu$. This tells us that we can substitute two nearby spinor indices (one undotted and one dotted) with a Lorentz index $\mu$ by contracting them with the Pauli matrices. In our example, the bispinor becomes a 4-vector $Q_\mu$. We can also deduce that $(\sigma_\mu)_{\alpha\dot{\alpha}}$ are the “natural spinor indices” of the Pauli matrices. Since the Pauli matrices $\sigma_\mu$ describe a basis of our internal (not spacetime) degrees of freedom, we are free to assume that they do not change under the action of the full Lorentz group $O(1,3)$:

$$\sigma_{\mu\alpha\dot{\alpha}} \rightarrow \sigma_{\mu\alpha\dot{\alpha}} = \Lambda^T_{\mu} \nu M_{\alpha}^{\beta} \bar{M}_{\dot{\alpha}}^{\dot{\beta}} \sigma_{\nu\beta\dot{\beta}} = \Lambda^{-1}_{\mu} \nu M_{\alpha}^{\beta} \bar{M}_{\dot{\alpha}}^{\dot{\beta}} \sigma_{\nu\beta\dot{\beta}}, \hspace{1cm} (53)$$

or, equivalently,

$$\Lambda^T_{\mu} \nu \sigma_{\nu\alpha\dot{\alpha}} = \Lambda^T_{\mu} \nu \bar{M}_{\alpha}^{\beta} \bar{M}_{\dot{\alpha}}^{\dot{\beta}} \sigma_{\nu\beta\dot{\beta}} = (M \sigma_\mu M^\dagger)_{\nu\mu\alpha\dot{\alpha}}. \hspace{1cm} (54)$$

3.7 The adjoint representation and the field representation

The commutation relations (7) can be written in terms of the structure constants as

$$[L_{\mu\nu}, L_{\rho\kappa}] = if_{\mu\nu\rho}^{\kappa\lambda} L_{\alpha\lambda}. \hspace{1cm} (55)$$

By means of the Jacobi identity, one can show that $-if_{\mu\nu\rho}^{\kappa\lambda}$ satisfy the commutation relations (7), carrying therefore a representation, called adjoint representation. Its dimension is given by the number of generators of $SL(2, \mathbb{C})$, which corresponds to 6. Until now, we only dealt with constant Grassmann numbers $\psi_\alpha$, but in general, they are functions of
an inner product

\[ SU \]

Therefore, the mixed delta tensor is also an operator, which turns out to be the unity matrix. Therefore, we can label the irreducible representation by the eigenvalue of the Casimir

\[ \Lambda \in SO^+(1,3) \]

let us say \( L \). One can easily show that \( \Lambda L \) commutes with every element of the algebra. Therefore it is possible to diagonalize simultaneously the Casimir operator together with, let us say \( L \). By Schur’s lemma, the Casimir operator for an irreducible representations is proportional to the unity matrix. Therefore, we can label the irreducible representation by the eigenvalue of the Casimir operator, which turns out to be \( n(n+1) \) with \( n = \frac{N}{2} \) being the eigenvalue of \( L \). In particular, the dimension of the representation is given by \( 2n + 1 \).

4 The special unitary group

The special unitary group \( SU(2) \) – the set of \( 2 \times 2 \) unitary matrices with unit determinant – is the double cover of the special orthogonal group \( SO(3) \), the group of spatial rotations in 3 dimensions. Since \( SU(2) \) is a subgroup of \( SL(2, \mathbb{C}) \), the delta and the epsilon tensors are also invariants of the former. One might expect that there exists at least one more invariant tensor. Indeed, this is the case. Consider a delta tensor with mixed indices \( \delta_{\alpha\beta} \), which, according to (41), transforms under \( U \in SU(2) \) (hence also \( \bar{U} \in SU(2) \)) as

\[ \delta_{\alpha\beta} \rightarrow \delta'_{\alpha\beta} = U_{\alpha}^\delta \delta_{\beta\delta} (U^T)_{\delta\alpha} = U_{\alpha}^\delta \delta_{\beta\delta} (U^{-1})_{\delta\alpha} = U_{\alpha}^\delta \delta_{\beta\delta} (U^{-1})_{\delta\alpha} = \delta_{\alpha\beta}. \]

Therefore, the mixed delta tensor is also an \( SU(2) \)-invariant tensor. It allows us to lower a dotted index to a undotted one. By introducing the inverse of the mixed delta tensor \( \delta_{\alpha\beta} \), we can also raise an undotted index to a dotted one. Furthermore, it allows us to contract dotted with undotted indices. Let us introduce an inner product

\[ (\bar{\phi}, \psi) := \bar{\phi}_\alpha \delta_{\alpha\beta} \psi_\beta. \]

As expected, the above inner product transforms invariantly under \( U \in SU(2) \):

\[ (\bar{\phi}, \psi) \rightarrow (\bar{\phi}', \psi') = \bar{\phi}_\alpha' \delta_{\alpha\beta} \psi_\beta = \bar{\phi}_\alpha' (U^T)_{\alpha\beta} \psi_\beta = \bar{\phi}_\alpha' \delta_{\alpha\beta} = (\bar{\phi}, \psi). \]

Rearranging the invariance of the mixed delta tensor yields

\[ (U^{-1T})_{\beta\beta} = \delta_{\alpha\beta} U_{\alpha}^\beta \delta_{\beta\alpha}, \]

or, equivalently,

\[ U^{-1T} = \delta^{-1} U \delta. \]

This restates the unitarity condition of \( SU(2) \). Hence, there are no other invariant tensors constraining further the group. In addition, we conclude that in \( SU(2) \), the fundamental representations are all equivalent and hence it suffices to consider only one of them, namely the Pauli spinor \( \psi_\alpha \) of non-relativistic quantum mechanics.

The Lie algebra \( \mathfrak{su}(2) \) is spanned by the generators \( L_i, i = 1, 2, 3 \), satisfying the commutation relations

\[ [L_i, L_j] = i\epsilon_{ijk} L_k. \]
4.1 The relation to the special linear group

As we already mentioned, comparing (16) with (66), one notes that the former corresponds to two disjoint sets of generators of $SU(2)$. Therefore, the identity

$$SL(2, \mathbb{C}) \cong SU(2)_+ \times SU(2)_-,$$

(67)

holds at least locally. It allows us to label an $SL(2, \mathbb{C})$ representation by a pair $(n, m)$, corresponding to, respectively, the eigenvalue of $J^+_3$ and $J^-_3$. Since $J_3 = J^+_3 + J^-_3$, the total spin of the representation is given by $n + m$ and its dimension is therefore $(2n + 1)(2m + 1)$. The inequivalent spin-$\frac{1}{2}$ representations are therefore given by

$$\psi_\alpha \equiv \begin{pmatrix} 1/2, 0 \end{pmatrix} \quad \text{and} \quad \bar{\psi}_\alpha \equiv \begin{pmatrix} 0, 1/2 \end{pmatrix},$$

(68)

describing for instance, respectively, the neutrino and the antineutrino. On the other hand, the bispinor introduced in section 3.6 carries the following representation:

$$Q_{\alpha\alpha} \equiv \begin{pmatrix} 1/2, 0 \end{pmatrix} \otimes \begin{pmatrix} 0, 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \ 1/2 \end{pmatrix},$$

(69)

corresponding to the vector representation of spin-1 (e.g. the electromagnetic potential). An example of tensor representation carrying a reducible representation is the rank $2 + 0$ tensor $Q_{\alpha\beta}$. Indeed,

$$Q_{\alpha\beta} \equiv \begin{pmatrix} 1/2, 0 \end{pmatrix} \otimes \begin{pmatrix} 1/2, 0 \end{pmatrix} = (0, 0) \oplus (1, 0),$$

(70)

is composed by the scalar (or trivial) representation of spin-0 (e.g. the Higgs boson) and by the self dual 2-form representation of spin-1. The antiself dual 2-form is denoted by $(0, 1)$ and a direct sum of the self dual with the antiself dual $(1, 0) \oplus (0, 1)$ gives the parity invariant 2-form field (e.g. the electromagnetic field strength). Other notable representations are given by $(1, 1)$, the traceless symmetric tensor field of spin-2 (e.g. the graviton) and by $(1, 1/2) \oplus (1/2, 1)$, the Rarita Schwinger field of spin-3/2 (e.g. the hypothetical gravitino).

5 The Dirac spinor

As we have seen in section 3.4, when constructing 2-dimensional representations of $SL(2, \mathbb{C})$, one has always to consider two inequivalent representations, take $\psi_\alpha$ and $\bar{\psi}_\alpha$ for convenience. Therefore, let us introduce a 4-component spinor called Dirac spinor carrying the representation (which is completely reducible under $SL(2, \mathbb{C})$)

$$\Psi = \begin{pmatrix} \psi_\alpha \cr \bar{\psi}_\alpha \end{pmatrix} \equiv \begin{pmatrix} 1/2, 0 \end{pmatrix} \oplus \begin{pmatrix} 0, 1/2 \end{pmatrix}.$$

(71)

5.1 The Clifford algebra

The Clifford algebra $Cl(1, 3; \mathbb{C})$ is given by those C-valued matrices $\gamma_\mu$ satisfying the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{I},$$

(72)

It turns out that the lowest dimensional representation of the above algebra is given by the $4 \times 4$ matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu\alpha \beta \\ \sigma_\mu\beta \alpha & 0 \end{pmatrix},$$

(73)

where

$$\bar{\sigma}_\mu^\alpha\beta = (\sigma_\mu^\alpha\beta)^* = -\epsilon^{\alpha\mu\gamma\delta}\sigma_\mu^\gamma\beta e^{\delta\beta} = \epsilon \sigma_\mu e^{-1})^\alpha\beta = (\epsilon \sigma_\mu e^{-1})^\alpha\beta = (\mathbb{I}, -\sigma)^\alpha\beta$$

(74)

is the complex conjugate of the Pauli matrices. It is clear that any similarity transformation preserves the Clifford algebra (72), here we have chosen the so-called Weyl basis to describe the $\gamma_\mu$’s. What does the Clifford algebra have to do with the Lorentz group and the Dirac spinor? First, observe that

$$S_{\mu\nu} := \frac{i}{4} [\gamma_\mu, \gamma_\nu] = \frac{i}{4} \begin{pmatrix} (\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu)^\alpha\beta & 0 \\ 0 & (\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu)^\alpha\beta \end{pmatrix}$$

(75)
obey the commutation relations (7), hence, it gives a representation of the \( \mathfrak{sl}(2, \mathbb{C}) \) algebra. Second, (75) is block diagonal. Let us consider the upper left block. By direct computation, one recovers exactly the generators \( L_{\mu\nu} \) of \( \mathfrak{sl}(2, \mathbb{C}) \) after having made the identification (14) with the plus sign in \( K_i \). That means that the upper left block acts on a spinor \( \psi_L \) generators is block diagonal. Let us consider the upper left block. By direct computation, one recovers exactly the

The above inner product is defined such that it is invariant under Lorentz transformations:

\[
\langle \Phi, \Psi \rangle_D := \bar{\Phi}\Psi = \Phi^\dagger \gamma_0 \Psi.
\]

The above inner product is defined such that it is invariant under Lorentz transformations:

\[
\langle \Phi, \Psi \rangle_D \to \langle \Phi', \Psi' \rangle_D = \Phi^\dagger \gamma_0 \exp[-i\frac{1}{2} \omega^{\mu\nu} \gamma_S] \gamma_0 \exp[i\frac{1}{2} \omega^{\mu\nu} S_{\mu\nu}] \Psi = \Phi^\dagger \gamma_0 \exp[-i\frac{1}{2} \omega^{\mu\nu} \gamma_S] \gamma_0 \exp[i\frac{1}{2} \omega^{\mu\nu} S_{\mu\nu}] \Psi = \Phi^\dagger \gamma_0 \Psi = \langle \Phi, \Psi \rangle_D.
\]

\( 1 \) We set \( \hbar = c = 1 \).