Spinors in 1+3 dimensions

Contents

1 Lie groups and Lie algebras

A *Lie group* G is a group which is also a differentiable manifold where the group operations of multiplication and inversion are smooth maps. To every Lie group G we can associate a Lie algebra $\mathfrak{g}$ whose elements lie in the tangent space at the identity element of the Lie group G. Because of the smooth structure of G, there exists a mapping from $\mathfrak{g}$ to G, namely the *exponential map*

$$g = \exp[\mathrm{i}\alpha^i T_i], \tag{1}$$

where $g \in G$, $T_i \in \mathfrak{g}$ and the α^i's are $\mathbb{R}$-valued parameters. However, note that the map does not have to be either injective or surjective. The T_i's can be chosen to form a basis of the algebra $\mathfrak{g}$ and are called *generators* of G.

1.1 Homomorphisms

A *(group) homomorphism* is a map Φ from a group G to another group G' which preserves the group operation, i.e., for any $g, h \in G$ one has $\Phi(gh) = \Phi(g)\Phi(h)$. Moreover, a bijective homomorphism is called *isomorphism*. A particular type of homomorphisms are the *(linear) representations* $\mathcal{R}_n\{G\}$ of G, that is, the homomorphisms mapping G to a subgroup of the *general linear group* $GL(n, \mathbb{K})$, the set of all invertible $n \times n$ matrices on a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, the particular representation acts on an n-dimensional vector space V_n over $\mathbb{K}$. Note that the commutation relations of the generators T_i of G:

$$[T_i, T_j] = \mathrm{i} f_{ij}^{\ \ k} T_k, \tag{2}$$

where f_{ijk} are called *structure constants*, are independent on the chosen representation and define therefore completely the Lie algebra $\mathfrak{g}$ associated to G. By the exponential map (1), one could obtain representations of the Lie group G by finding representations of the associated Lie algebra $\mathfrak{g}$.
Take a representation $\mathcal{R}_n\{G\}$ which acts on an n-dimensional vector space V_n. If, by acting on any element

of the subspace W_m with any group element in the representation $\mathcal{R}_n\{G\}$, the resulting element is still in W_m, then the subspace W_m is called an invariant subspace of V_n. If the only invariant subspaces of V_n are given by the zero vector space $\{0\}$ and the vector space itself V_n, then the representation $\mathcal{R}_n\{G\}$ is called an *irreducible representation*.
A *completely reducible* representation is a representation which can be written as a direct sum of irreducible representations, i.e., for any $g \in G$, $\mathcal{R}_n(g)$ can be written in block diagonal form where any block relates to an irreducible representation. Note however that, if a representation is reducible, it does not mean that it is also completely reducible. *Unitary representations* (where $\mathcal{R}_n^\dagger(g) = \mathcal{R}_n^{-1}(g)$ for all $g \in G$) are always completely reducible. On the other hand, for any reducible representation $\mathcal{R}_n\{G\}$, $\mathcal{R}_n(g)$ can at least be written as an upper triangular block matrix.

2 The Lorentz group

An example of a Lie group is the *Lorentz group* $O(1,3)$, the manifold of $\mathbb{R}$-valued 4×4 matrices $\Lambda^\mu{}_\nu$ which are orthogonal, that is, they leave the Minkowski metric $\eta = \mathrm{diag}(+,-,-,-)$ invariant:

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \Lambda^{T}{}_\mu^\rho\eta_{\rho\sigma}\Lambda^\sigma{}_\nu = (\Lambda^T\eta\Lambda)_{\mu\nu}. \tag{3}$$

The *proper, orthochronous Lorentz group* in 1+3 dimensions $SO^+(1,3)$ is the connected submanifold of $O(1,3)$ containing the identity transformation. It can be characterized as the set of all the orthogonal $\mathbb{R}$-valued 4×4 matrices with unit determinant and with $\Lambda^0{}_0 \geq 1$. An infinitesimal Lorentz transformation acts as

$$\Lambda^\mu{}_\nu x^\nu \simeq x^\mu + \omega^\mu{}_\nu x^\nu, \tag{4}$$

where $\omega^{\rho\sigma}$ is an $\mathbb{R}$-valued tensor. By means of (3), one gets

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu \simeq \eta_{\rho\sigma}(\delta^\rho{}_\mu + \omega^\rho{}_\mu)(\delta^\sigma{}_\nu + \omega^\sigma{}_\nu) = \eta_{\mu\nu} + \omega^\mu{}_\nu + \omega^\nu{}_\mu \equiv \eta_{\mu\nu}. \tag{5}$$

Therefore, $\omega^{\mu\nu}$ has to be antisymmetric. The Lie algebra $\mathfrak{so}(1,3)$ of $SO^+(1,3)$ is spanned by the six generators $J_i, K_i \in \mathfrak{so}(1,3)$, $i = 1,2,3$, associated, respectively, to spatial rotations and boosts. The commutation relations of the generators are the following:

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \qquad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k. \tag{6}$$

One can redefine the 6 generators J_i, K_i by making use of the fact that a 4×4 antisymmetric matrix $L_{\mu\nu} = -L_{\nu\mu}$ has exactly 6 independent entries. By setting $L_{ij} = \epsilon_{ijk}J_k$ and $L_{i0} = K_i$, the commutation relations (6) become

$$\boxed{[L_{\mu\nu}, L_{\rho\sigma}] = \mathrm{i}\eta_{\mu\sigma}L_{\nu\rho} + \mathrm{i}\eta_{\nu\rho}L_{\mu\sigma} - \mathrm{i}\eta_{\mu\rho}L_{\nu\sigma} - \mathrm{i}\eta_{\nu\sigma}L_{\mu\rho}.} \tag{7}$$

In complete analogy to (1), the exponential map in the defining representation can be written as

$$\boxed{\Lambda^\mu{}_\nu = \exp[\frac{\mathrm{i}}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu].} \tag{8}$$

Note that, owing to the reality of $\Lambda^\mu{}_\nu$, the generators in the defining representation $(L_{\rho\sigma})^\mu{}_\nu$ have to be imaginary. Putting together (4) and (8) and expanding up to first order in $\omega^\mu{}_\nu$ yields

$$x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \frac{1}{2}\left(\delta^\mu{}_\rho\eta_{\sigma\nu} - \delta^\mu{}_\sigma\eta_{\rho\nu}\right)\omega^{\rho\sigma}x^\nu \equiv x^\mu + \frac{\mathrm{i}}{2}\omega^{\rho\sigma}(L_{\rho\sigma})^\mu{}_\nu. \tag{9}$$

Therefore, we can identify

$$(L_{\rho\sigma})^\mu{}_\nu = -\mathrm{i}\left(\delta^\mu{}_\rho\eta_{\sigma\nu} - \delta^\mu{}_\sigma\eta_{\rho\nu}\right), \tag{10}$$

which are the generators of the Lorentz group in the defining representation. Indeed, (10) satifies the Lorentz algebra given in (7).

3 The special linear group

The Lorentz group $SO^+(1,3)$ has the *special linear group* $SL(2,\mathbb{C})$ as its double cover, i.e. there exists a double valued surjective homomorphism $\Phi : SL(2,\mathbb{C}) \to SO^+(1,3)$. For any 4-vector x^μ and $M \in SL(2,\mathbb{C})$, define

$$y^\mu\sigma_\mu := Mx^\nu\sigma_\nu M^\dagger, \tag{11}$$

where $M^\dagger$ is the adjoint matrix of M and

$$\sigma_\mu = (\mathbb{I}, \vec{\sigma}) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{12}$$

are the *generalized Pauli matrices*. It is then possible to show that y^μ corresponds to a Lorentz transformed 4-vector and that every element of $SO^+(1,3)$ is related to two elements of $SL(2,\mathbb{C})$, namely M and $-M$, by a homomorphism. Due to the covering, the two groups are locally (in a neighborhood of the identity element) isomorphic, that is, $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$. Therefore, the generators of $SL(2,\mathbb{C})$ also satisfy the commutation relations (6) and (7). Indeed, taking into account (1), one gets

$$1 = \det M = \det \exp[\mathrm{i}(\alpha^i J_i + \beta^j K_j)] = \exp[\mathrm{i}\ \mathrm{tr}(\alpha^i J_i + \beta^j K_j)]. \tag{13}$$

Since the $\mathbb{R}$-valued parameters α^i, β^j are arbitrary, the generators of $SL(2,\mathbb{C})$ have to be traceless. A basis for the space of traceless complex 2×2 matrices is given by the hermitian Pauli matrices σ_i and the antihermitian Pauli matrices $\mathrm{i}\sigma_i$. By setting

$$J_i = \frac{\sigma_i}{2} \quad \text{and} \quad K_i = \pm \mathrm{i}\frac{\sigma_i}{2}, \tag{14}$$

those clearly satisfy the commutation relations (6). The circumstance that we can not choose all generators to be hermitian derives from the fact that $SL(2,\mathbb{C})$, as well as the Lorentz group, is not topologically compact. Let us introduce a change of basis defined by

$$J_i^\pm := \frac{1}{2}\left(J_i \pm \mathrm{i}K_i\right). \tag{15}$$

Then, the commutation relations (6) become

$$[J_i^\pm, J_j^\pm] = \mathrm{i}\epsilon_{ijk} J_k^\pm, \qquad [J_i^\pm, J_j^\mp] = 0, \tag{16}$$

which, as we will see in section 4.1, resemble two disjoint sets of generators of $SU(2)$.

3.1 The fundamental representation

The *fundamental representation* of $SL(2,\mathbb{C})$ is given by 2×2 matrices with unit determinant. Those matrices act naturally on a $\mathbb{C}$-valued 2-dimensional object ψ_α , $\alpha = 1, 2$ which we call a *(left-handed Weyl) spinor*. Take $M \in SL(2,\mathbb{C})$, then the transformation rule is defined as

$$\boxed{\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta.} \tag{17}$$

Throughout this survey, consider that the positions of the spinor indices are of vital importance. Moreover, bear in mind that the "natural way" to contract spinor indices is from upper left to lower right. The reason will become clear in section 3.3.

3.2 The dual representation

This section aims to introduce the *dual representation* and to show that the fundamental representation is equivalent to the former, i.e., there exists a similarity transformation which maps M to M^{-1T}. If $M \in SL(2,\mathbb{C})$, then also M^{-1T} forms a representation of $SL(2,\mathbb{C})$. Indeed, take $M, N, O \in SL(2,\mathbb{C})$ such that $MN = O$, then

$$M^{-1T} N^{-1T} = (N^{-1}M^{-1})^T = (MN)^{-1T} = O^{-1T}. \tag{18}$$

This representation acts on another kind of spinor, the dual (Weyl) spinor ψ^α, which transforms under an $M \in SL(2,\mathbb{C})$ as

$$\boxed{\psi^\alpha \to \psi'^\alpha = (M^{-1T})^\alpha{}_\beta \psi^\beta = \psi^\beta (M^{-1})_\beta{}^\alpha.} \tag{19}$$

It is now possible to define an inner product

$$\boxed{\langle \phi, \psi \rangle \equiv \phi\psi := \phi^\alpha \psi_\alpha,} \tag{20}$$

which is $SL(2,\mathbb{C})$-invariant:

$$\phi^\alpha \psi_\alpha \to \phi'^\alpha \psi'_\alpha = \phi^\beta (M^{-1})_\beta{}^\alpha M_\alpha{}^\gamma \psi_\gamma = \phi^\alpha \psi_\alpha, \tag{21}$$

motivating the label "dual" for ϕ^α. Remark that, as we will see later, $\phi\psi \neq \phi_\alpha\psi^\alpha$. What are the invariant tensors under the $SL(2,\mathbb{C})$ group? A (trivially) invariant tensor is the delta tensor $\delta = \text{diag}(+,+)$ as one can see from

$$\delta_\alpha^{\ \beta} \to \delta_\alpha^{\prime\ \beta} = M_\alpha^{\ \gamma}\delta_\gamma^{\ \delta}(M^{-1})_\delta^{\ \beta} = M_\alpha^{\ \gamma}(M^{-1})_\gamma^{\ \beta} = \delta_\alpha^{\ \beta}. \tag{22}$$

Let us now introduce the *epsilon tensor*

$$\boxed{\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha},\ \epsilon_{12} := 1.} \tag{23}$$

Furthermore, denote the inverse of the epsilon tensor with $\epsilon^{\alpha\beta}$, fixed by the relation

$$\boxed{\epsilon^{\beta\gamma}\epsilon_{\gamma\alpha} = \delta^\beta_{\ \alpha} \qquad \text{with} \qquad \delta_\alpha^{\ \beta}\psi_\beta = \psi_\alpha.} \tag{24}$$

However, as we will see in section 3.3, $\delta_\alpha^{\ \beta} = -\delta^\beta_{\ \alpha}$. By inspection, one can check that such an object exists and is given by

$$\boxed{\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha},\ \epsilon^{12} = 1.} \tag{25}$$

For $M \in SL(2,\mathbb{C})$, $\epsilon_{\alpha\beta}$ transforms as

$$\epsilon_{\alpha\beta} \to \epsilon'_{\alpha\beta} = M_\alpha^{\ \gamma}M_\beta^{\ \delta}\epsilon_{\gamma\delta}. \tag{26}$$

It is easy to see that also $\epsilon'_{\alpha\beta}$ is antisymmetric and has therefore to be a multiple of the epsilon tensor, i.e. $\epsilon'_{\alpha\beta} = \eta\ \epsilon_{\alpha\beta}$. By evaluating the latter formula for $\alpha = 1, \beta = 2$, one gets

$$\eta = \eta\ \epsilon_{12} = M_1^{\ 1}M_2^{\ 2}\epsilon_{12} + M_1^{\ 2}M_2^{\ 1}\epsilon_{21} = M_1^{\ 1}M_2^{\ 2} - M_1^{\ 2}M_2^{\ 1} = \det[M] = 1. \tag{27}$$

We conclude that the epsilon tensor is an invariant tensor under $SL(2,\mathbb{C})$ transformations (a similar analysis applied to $\epsilon^{\alpha\beta}$ yields also invariance under $SL(2,\mathbb{C})$ of the latter). Consider, for any $N \in GL(2,\mathbb{C})$, the identity

$$\det(N)\epsilon_{\alpha\beta} = N_\alpha^{\ \gamma}N_\beta^{\ \delta}\epsilon_{\gamma\delta}. \tag{28}$$

Therefore, we can equivalently define $SL(2,\mathbb{C})$ as the set of all 2×2 matrices which leave the epsilon tensor invariant. Now assume the existence of a non-antisymmetric $SL(2,\mathbb{C})$-invariant tensor $\rho_{\alpha\beta}$, that is,

$$\rho_{\alpha\beta} = M_\alpha^{\ \gamma}M_\beta^{\ \delta}\rho_{\gamma\delta} \qquad \text{with} \qquad \rho_{\alpha\beta} \neq -\rho_{\beta\alpha} \tag{29}$$

for any $M \in SL(2,\mathbb{C})$. This would imply another constraint on the matrices M generating a true subgroup of $SL(2,\mathbb{C})$. We conclude that, up to a scalar prefactor, the ϵ's are the only non-trivially invariant tensors of rank two of the double cover of the Lorentz group. One can restate the invariance of the epsilon tensor as

$$(M^{-1T})^\alpha_{\ \ \beta} = -\epsilon^{\alpha\gamma}M_\gamma^{\ \delta}\epsilon_{\delta\beta}, \tag{30}$$

or, shorthand,

$$M^{-1T} = \epsilon^{-1}M\epsilon. \tag{31}$$

This is the aforementioned similarity transformation which describes an isomorphism relating the fundamental with the dual representation.

3.3 Handling spinor indices

Let us examine how the contraction $\psi^\beta\epsilon_{\beta\alpha}$ transforms under $M \in SL(2,\mathbb{C})$:

$$\psi^\beta\epsilon_{\beta\alpha} \to \psi'^\beta\epsilon'_{\beta\alpha} = (M^{-1T})^\beta_{\ \ \epsilon}\psi^\epsilon M_\beta^{\ \gamma}M_\alpha^{\ \delta}\epsilon_{\gamma\delta} = \psi^\epsilon\delta^\gamma_\epsilon M_\alpha^{\ \delta}\epsilon_{\gamma\delta} = M_\alpha^{\ \delta}\ \psi^\gamma\epsilon_{\gamma\delta}, \tag{32}$$

i.e., the contraction transforms as a spinor. Due to the invariance of the epsilon tensor, note that $\psi'^\beta\epsilon_{\beta\alpha} = \psi'^\beta\epsilon'_{\beta\alpha}$. That is, the transformed spinor corresponds to the contraction with the beforehand transformed dual spinor. Therefore, we can identify

$$\boxed{\psi_\alpha \equiv \psi^\beta\epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta}\psi^\beta.} \tag{33}$$

As we have seen, we can use the epsilon tensor to lower spinor indices. It would be useful to find an object $\epsilon^{\alpha\beta}$ capable of raising the spinor indices:

$$\boxed{\psi^\alpha := \epsilon^{\alpha\beta}\psi_\beta.} \tag{34}$$

First, note that

$$\epsilon_{\alpha\beta} = \epsilon^{\gamma\delta}\epsilon_{\gamma\alpha}\epsilon_{\delta\beta} = \delta_{\alpha}{}^{\delta}\epsilon_{\delta\beta} = \epsilon_{\alpha\beta}, \tag{35}$$

i.e., $\epsilon^{\alpha\beta}$ can be recovered by raising the indices of $\epsilon_{\alpha\beta}$ and vice versa. For consistency, let us verify if lowering and subsequently raising a spinor index corresponds to the identity operator. Indeed, by using equation (24) one gets

$$\psi_\alpha = \psi^\gamma\epsilon_{\gamma\alpha} = \epsilon^{\gamma\beta}\psi_\beta\epsilon_{\gamma\alpha} = \delta_\alpha{}^\beta\psi_\beta = \psi_\alpha. \tag{36}$$

One can easily check that also the converse is true, i.e., raising and subsequently lowering leads to the same dual spinor. Observe that if we assume $\delta^\beta{}_\alpha = \delta_\alpha{}^\beta$ then one has

$$\psi_\alpha = \psi^\gamma\epsilon_{\gamma\alpha} = \epsilon^{\gamma\beta}\psi_\beta\epsilon_{\gamma\alpha} = -\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta}\psi_\beta = -\delta_\alpha{}^\beta\psi_\beta = -\psi_\alpha. \tag{37}$$

This is a pretty awkward contradiction. However, a way to bypass this issue is easily found. So far we contracted only from upper left to lower right, the introduction of a contraction from lower left to upper right requires that a minus sign pops out in the calculation. This gives for example

$$\boxed{\delta^\beta{}_\alpha = \epsilon^{\beta\gamma}\epsilon_{\gamma\alpha} = \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta_\alpha{}^\beta} \tag{38}$$

and

$$\boxed{\phi^\alpha\psi_\alpha = -\phi_\alpha\psi^\alpha.} \tag{39}$$

3.4 The antifundamental representation and its dual

The fundamental and its dual representation are not the only (up to similarity transformations) 2-dimensional representations of $SL(2,\mathbb{C})$. For $M \in SL(2,\mathbb{C})$, define the complex conjugation as

$$\boxed{\bar{M}_{\dot\alpha}{}^{\dot\beta} := (M_\alpha{}^\beta)^*,} \tag{40}$$

but with the convention that the "natural way" of contracting dotted spinor indices is, contrary to the undotted ones, from lower left to upper right. This gives the *antifundamental representation* of the double cover of the Lorentz group. However, be aware that the definition (40) might be different in some textbooks. The dots on the indices are introduced merely as a mnemonic device to distinguish the fundamental from the antifundamental representation, in order to avoid, for example, contractions between dotted and undotted indices. Analogously to the above, $\bar{M}^{-1T}$ defines the *dual of the antifundamental representation*. According to the definitions given in (17) and (19), those representations act on *(right-handed Weyl) spinors* (with bars on top) in the following way:

$$\boxed{\bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = \bar\psi_{\dot\beta}(M^\dagger)^{\dot\beta}{}_{\dot\alpha} \quad \text{and} \quad \bar\psi^{\dot\alpha} \to \bar\psi'^{\dot\alpha} = (M^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta}.} \tag{41}$$

Therefore, we can identify

$$\boxed{\bar\psi_{\dot\alpha} = (\psi_\alpha)^*.} \tag{42}$$

The complex conjugate of the epsilon tensor is given by

$$\boxed{(\epsilon_{\alpha\beta})^* = \bar\epsilon_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta},} \tag{43}$$

and equivalently for its inverse. We see that, bearing in mind the adopted conventions, all the discussions made for the fundamental representation and its dual apply also to the antifundamental representation and its dual. In particular, remark that equations (33), (34) and (38) become, respectively,

$$\boxed{\bar\psi_{\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}, \quad \bar\psi^{\dot\alpha} = -\bar\psi_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}, \quad \bar\psi_{\dot\alpha} = \bar\psi_{\dot\beta}\delta^{\dot\beta}{}_{\dot\alpha} = -\delta_{\dot\alpha}{}^{\dot\beta}\bar\psi_{\dot\beta} = \bar\psi_{\dot\alpha}.} \tag{44}$$

Additionally, by using (30) one gets

$$(\bar{M}^{-1T})^{\dot\alpha}{}_{\dot\beta} = [(M^{-1T})^\alpha{}_\beta]^* = -(\epsilon^{\alpha\gamma}M_\gamma{}^\delta\epsilon_{\delta\beta})^* = -\epsilon^{\dot\alpha\dot\gamma}\bar{M}_{\dot\gamma}{}^{\dot\delta}\epsilon_{\dot\delta\dot\beta} = (\epsilon^{-1}\bar{M}\epsilon)^{\dot\alpha}{}_{\dot\beta}. \tag{45}$$

Therefore, the latter two representations are equivalent to one another, but not equivalent to the former two since there does not exist an invariant tensor which relates the representations by a similarity transformation. When considering fundamental representations of $SL(2,\mathbb{C})$, one has to deal with two inequivalent representations.

3.5 Grassmann numbers

Let us commute the spinors in the inner product introduced in (20):

$$\phi\psi = \phi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\phi_\beta\psi_\alpha = -\epsilon^{\alpha\beta}\psi_\beta\phi_\alpha = -\psi^\alpha\phi_\alpha = -\psi\phi \tag{46}$$

and equivalently $\bar{\phi}\bar{\psi} = \bar{\phi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = -\bar{\psi}\bar{\phi}$. Everything so far has been classical and therefore we treated the entries of the spinors as *c*-numbers (c stands for "commuting"). When going to the quantum mechanical regime, one promotes the spinors to fermionic operators, i.e., operators satisfying the *anticommutation relations*

$$\boxed{\{\psi_\alpha, \phi_\beta\} = \{\psi_\alpha, \bar{\phi}_{\dot{\beta}}\} = \{\bar{\psi}_{\dot{\alpha}}, \bar{\phi}_{\dot{\beta}}\} = 0} \tag{47}$$

but which are still commuting with *c*-numbers. In particular, it follows that $\psi_1\psi_1 = \psi_2\psi_2 = 0$ and etc.. The anticommutation relations in (47) define a *Grassmann algebra* and the entries of the spinors are therefore called *Grassmann numbers*. Equation (46) now becomes

$$\boxed{\phi\psi = \phi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\phi_\beta\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\phi_\alpha = \psi^\alpha\phi_\alpha = \psi\phi, \qquad \bar{\phi}\bar{\psi} = \bar{\psi}\bar{\phi}.} \tag{48}$$

Under complex conjugation we set the rule that the Grassmann numbers interchange:

$$\boxed{(\phi_\alpha\psi_\beta)^* = (\psi_\beta)^*(\phi_\alpha)^* = \bar{\psi}^{\dot{\beta}}\bar{\phi}^{\dot{\alpha}}} \tag{49}$$

such that

$$\boxed{(\phi\psi)^* = (\phi^\alpha\psi_\alpha)^* = (\psi^\alpha)^*(\phi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = \bar{\psi}\bar{\phi} = \bar{\phi}\bar{\psi}.} \tag{50}$$

3.6 Tensorial representations

A tensorial representation of rank $s+t$ of $SL(2,\mathbb{C})$ is given by a tensor $Q_{\alpha_1\ldots\alpha_s\dot{\alpha}_1\ldots\dot{\alpha}_t}$. As we will see later, tensorial representations allow for formulating representations for any spin $s \in \frac{\mathbb{N}_0}{2}$. However, general tensorial representations are reducible. To make them irreducible, one has to impose a bunch of relations on the tensor Q. A didactical example of tensorial representation is given by the bispinor $Q_{\alpha\dot{\alpha}}$, a tensor of rank $1+1$. It transforms according to

$$Q_{\alpha\dot{\alpha}} \to Q'_{\alpha\dot{\alpha}} = M_\alpha{}^\beta Q_{\beta\dot{\beta}}(\bar{M}^T)^{\dot{\beta}}{}_{\dot{\alpha}} = M_\alpha{}^\beta Q_{\beta\dot{\beta}}(M^\dagger)^{\dot{\beta}}{}_{\dot{\alpha}}, \tag{51}$$

that is,

$$Q \to Q' = MQM^\dagger. \tag{52}$$

By means of equation (11), this corresponds to a Lorentz transformation. Therefore, the bispinor is nothing other than $Q = x^\mu\sigma_\mu$ for a specific 4-vector x^μ. This tells us that we can substitute two nearby spinor indices (one undotted and one dotted) with a Lorentz index μ by contracting them with the Pauli matrices. In our example, the bispinor becomes a 4-vector Q_μ. We can also deduce that $(\sigma_\mu)_{\alpha\dot{\alpha}}$ are the "natural spinor indices" of the Pauli matrices.
Since the Pauli matrices σ_μ describe a basis of our internal (not spacetime) degrees of freedom, we are free to assume that they do not change under the action of the full Lorentz group $O(1,3)$:

$$\sigma_{\mu\alpha\dot{\alpha}} \to \sigma_{\mu\alpha\dot{\alpha}} = \Lambda^T{}_\mu{}^\nu M_\alpha{}^\beta \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}}\sigma_{\nu\beta\dot{\beta}} = \Lambda^{-1}{}_\mu{}^\nu M_\alpha{}^\beta \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}}\sigma_{\nu\beta\dot{\beta}}, \tag{53}$$

or, equivalently,

$$\boxed{\Lambda_\mu{}^\nu\sigma_{\nu\alpha\dot{\alpha}} = M_\alpha{}^\beta\bar{M}_{\dot{\alpha}}{}^{\dot{\beta}}\sigma_{\mu\beta\dot{\beta}} = (M\sigma_\mu M^\dagger)_{\mu\alpha\dot{\alpha}}.} \tag{54}$$

3.7 The adjoint representation and the field representation

The commutation relations (7) can be written in terms of the structure constants as

$$[L_{\mu\nu}, L_{\rho\sigma}] = \mathrm{i}f_{\mu\nu\rho\sigma}{}^{\kappa\lambda}L_{\kappa\lambda}. \tag{55}$$

By means of the *Jacobi identity*, one can show that $-\mathrm{i}(f_{\mu\nu})_{\rho\sigma}{}^{\kappa\lambda}$ satisfy the commutation relations (7), carrying therefore a representation, called *adjoint representation*. Its dimension is given by the number of generators of $SL(2,\mathbb{C})$, which corresponds to 6.
Until now, we only dealt with constant Grassmann numbers ψ_α, but in general, they are functions of

spacetime. For instance, an active transformation of the left-handed Weyl spinor function with respect to $\Lambda \in SO^+(1,3)$ reads

$$\psi_\alpha(x) \to \psi'_\alpha(x) = M_\alpha^{\ \beta}(\Lambda)\psi_\beta(\Lambda^{-1}x). \tag{56}$$

For the moment let us forget about the action of M on ψ_α. Taylor expanding the right hand side of the above equation and making use of (4) yields

$$\psi_\alpha(\Lambda^{-1}x) \simeq \psi_\alpha(x) + (x^\mu - (x^\mu + \omega^\mu_{\ \nu}x^\nu))\partial_\mu\psi_\alpha(x) = \psi_\alpha(x) - \omega^\mu_{\ \nu}x^\nu\partial_\mu\psi_\alpha(x) \tag{57}$$

On the other hand, by means of the exponential map (8), one expects to have an antisymmetric generator $\Sigma_{\mu\nu}$ such that

$$\psi_\alpha(\Lambda^{-1}x) = \exp\left[\frac{\mathrm{i}}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\right]\psi_\alpha(x). \tag{58}$$

To first order in $\omega^\mu_{\ \nu}$, this gives the identification

$$-\omega^\mu_{\ \nu}x^\nu\partial_\mu = -\frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu) \equiv \frac{\mathrm{i}}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}. \tag{59}$$

Eventually, one gets the generators

$$\Sigma_{\mu\nu} = -\mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu), \tag{60}$$

which, as one expects, satisfy also the commutation relations (7). Since $\Sigma_{\mu\nu}$ acts on a space of functions which is an infinite-dimensional vector space, it corresponds to an infinite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, the *field representation.*

4 The special unitary group

The *special unitary group* $SU(2)$ – the set of 2×2 unitary matrices with unit determinant – is the double cover of the *special orthogonal group* $SO(3)$, the group of spatial rotations in 3 dimensions. Since $SU(2)$ is a subgroup of $SL(2,\mathbb{C})$, the delta and the epsilon tensors are also invariants of the former. One might expect that there exists at least one more invariant tensor. Indeed, this is the case. Consider a delta tensor with mixed indices $\delta_{\alpha\dot\alpha}$, which, according to (41), transforms under $U \in SU(2)$ (hence also $\bar U \in SU(2)$) as

$$\delta_{\alpha\dot\alpha} \to \delta'_{\alpha\dot\alpha} = U_\alpha^{\ \beta}\delta_{\beta\dot\beta}(\bar U^T)^{\dot\beta}_{\ \dot\alpha} = U_\alpha^{\ \beta}\delta_{\beta\dot\beta}(U^\dagger)^{\dot\beta}_{\ \dot\alpha} = U_\alpha^{\ \beta}\delta_{\beta\dot\beta}(U^{-1})^{\dot\beta}_{\ \dot\alpha} = \delta_{\alpha\dot\alpha}. \tag{61}$$

Therefore, the mixed delta tensor is also an $SU(2)$-invariant tensor. It allows us to lower a dotted index to a undotted one. By introducing the inverse of the mixed delta tensor $\delta^{\dot\alpha\alpha}$, we can also raise an undotted index to a dotted one. Furthermore, it allows us to contract dotted with undotted indices. Let us introduce an inner product

$$\boxed{(\bar\phi,\psi) := \bar\phi_{\dot\alpha}\delta^{\dot\alpha\alpha}\psi_\alpha.} \tag{62}$$

As expected, the above inner product transforms invariantly under $U \in SU(2)$:

$$(\bar\phi,\psi) \to (\bar\phi',\psi') = \bar\phi'_{\dot\alpha}\delta^{\dot\alpha\alpha}\psi'_\alpha = \bar\phi_{\dot\beta}(\bar U^T)^{\dot\beta}_{\ \alpha}\delta^{\dot\alpha\alpha}U_\alpha^{\ \beta}\psi_\beta = \bar\phi_{\dot\beta}\delta^{\dot\beta\beta}\psi_\beta = (\bar\phi,\psi). \tag{63}$$

Rearranging the invariance of the mixed delta tensor yields

$$(\bar U^{-1T})^{\dot\alpha}_{\ \dot\beta} = \delta^{\dot\alpha\alpha}U_\alpha^{\ \beta}\delta_{\beta\dot\beta}, \tag{64}$$

or, equivalently,

$$\bar U^{-1T} = \delta^{-1}U\delta. \tag{65}$$

This restates the unitarity condition of $SU(2)$. Hence, there are no other invariant tensors constraining further the group. In addition, we conclude that in $SU(2)$, the fundamental representations are all equivalent and hence it suffices to consider only one of them, namely the *Pauli spinor* ψ_α of non-relativistic quantum mechanics.

The Lie algebra $\mathfrak{su}(2)$ is spanned by the generators $L_i, i = 1,2,3$, satisfying the commutation relations

$$\boxed{[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k.} \tag{66}$$

One can easily show that L_iL_i is a *Casimir operator* of $\mathfrak{su}(2)$, that is, L_iL_i commutes with every element of the algebra. Therefore it is possible to diagonalize simultaneously the Casimir operator together with, let us say L_3. By *Schur's lemma*, the Casimir operator for an irreducible representations is proportional to the unity matrix. Therefore, we can label the irreducible representation by the eigenvalue of the Casimir operator, which turns out to be $n(n+1)$ with $n = \frac{\mathbb{N}_0}{2}$ being the eigenvalue of L_3. In particular, the dimension of the representation is given by $2n+1$.

4.1 The relation to the special linear group

As we already mentioned, comparing (16) with (66), one notes that the former corresponds to two disjoint sets of generators of $SU(2)$. Therefore, the identity

$$\boxed{SL(2,\mathbb{C}) \cong SU(2)_+ \times SU(2)_-,} \tag{67}$$

holds at least locally. It allows us to label an $SL(2,\mathbb{C})$ representation by a pair (n,m), corresponding to, respectively, the eigenvalue of J_3^+ and J_3^-. Since $J_3 = J_3^+ + J_3^-$, the total spin of the representation is given by $n+m$ and its dimension is therefore $(2n+1)(2m+1)$. The inequivalent spin-$\frac{1}{2}$ representations are therefore given by

$$\boxed{\psi_\alpha \equiv \left(\frac{1}{2},0\right) \quad \text{and} \quad \bar{\psi}_{\dot{\alpha}} \equiv \left(0,\frac{1}{2}\right),} \tag{68}$$

describing for instance, respectively, the neutrino and the antineutrino. On the other hand, the bispinor introduced in section 3.6 carries the following representation:

$$Q_{\alpha\dot{\alpha}} \equiv \left(\frac{1}{2},0\right) \otimes \left(0,\frac{1}{2}\right) = \left(\frac{1}{2},\frac{1}{2}\right), \tag{69}$$

corresponding to the vector representation of spin-1 (e.g. the electromagnetic potential). An example of tensor representation carrying a reducible representation is the rank $2+0$ tensor $Q_{\alpha\beta}$. Indeed,

$$Q_{\alpha\beta} \equiv \left(\frac{1}{2},0\right) \otimes \left(\frac{1}{2},0\right) = (0,0) \oplus (1,0)\,, \tag{70}$$

is composed by the scalar (or trivial) representation of spin-0 (e.g. the Higgs boson) and by the *self dual 2-form* representation of spin-1. The *antiself dual 2-form* is denoted by $(0,1)$ and a direct sum of the self dual with the antiself dual $(1,0)\oplus(0,1)$ gives the *parity invariant 2-form field* (e.g. the electromagnetic field strength). Other notable representations are given by $(1,1)$, the *traceless symmetric tensor field* of spin-2 (e.g. the graviton) and by $(1,1/2)\oplus(1/2,1)$, the *Rarita Schwinger field* of spin-3/2 (e.g. the hypothetical gravitino).

5 The Dirac spinor

As we have seen in section 3.4, when constructing 2-dimensional representations of $SL(2,\mathbb{C})$, one has always to consider two inequivalent representations, take ψ_α and $\bar{\phi}^{\dot{\alpha}}$ for convenience. Therefore, let us introduce a 4-component spinor called *Dirac spinor* carrying the representation (which is completely reducible under $SL(2,\mathbb{C})$)

$$\boxed{\Psi = \begin{pmatrix}\psi_\alpha \\ \bar{\phi}^{\dot{\alpha}}\end{pmatrix} \equiv \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right).} \tag{71}$$

5.1 The Clifford algebra

The *Clifford algebra* $Cl(1,3;\mathbb{C})$ is given by those $\mathbb{C}$-valued matrices γ_μ satisfying the anticommutation relations

$$\boxed{\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{I}.} \tag{72}$$

It turns out that the lowest dimensional representation of the above algebra is given by the 4×4 matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot{\beta}} \\ \bar{\sigma}_\mu{}^{\dot{\alpha}\beta} & 0\end{pmatrix}, \tag{73}$$

where

$$\bar{\sigma}_\mu{}^{\dot{\alpha}\beta} = (\sigma_\mu{}^{\alpha\dot{\beta}})^* = -(\epsilon^{\alpha\gamma}\sigma_{\mu\gamma\dot{\delta}}\epsilon^{\dot{\delta}\dot{\beta}})^* = [(\epsilon\sigma_\mu\epsilon^{-1})^*]^{\dot{\alpha}\beta} = (\epsilon\sigma_\mu^*\epsilon^{-1})^{\dot{\alpha}\beta} = (\mathbb{I},-\vec{\sigma})^{\dot{\alpha}\beta} \tag{74}$$

is the complex conjugate of the Pauli matrices. It is clear that any similarity transformation preserves the Clifford algebra (72), here we have chosen the so-called *Weyl basis* to describe the γ_μ's. What does the Clifford algebra have to do with the Lorentz group and the Dirac spinor? First, observe that

$$S_{\mu\nu} := \frac{\mathrm{i}}{4}[\gamma_\mu,\gamma_\nu] = \frac{\mathrm{i}}{4}\begin{pmatrix}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\beta & 0 \\ 0 & (\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix} \tag{75}$$

obeys the commutation relations (7), hence, it gives a representation of the $\mathfrak{sl}(2,\mathbb{C})$ algebra. Second, (75) is block diagonal. Let us consider the upper left block. By direct computation, one recovers exactly the generators $L_{\mu\nu}$ of $\mathfrak{sl}(2,\mathbb{C})$ after having made the identification (14) with the plus sign in K_i. That means that the upper left block acts on a spinor ψ_β. On equal terms, one can show that the lower right block corresponds to $L_{\mu\nu}$ with the minus sign in K_i. Hence, the lower right block acts on the spinor $\bar{\phi}^{\dot{\alpha}}$. This can be seen by considering the exponential map of the fundamental representation:

$$\begin{aligned} M_\alpha{}^\beta &= \exp\left[\mathrm{i}(\alpha^i J_i + \beta^j K_j)\right]_\alpha{}^\beta = \left\{\exp\left[-\mathrm{i}(\alpha^i J_i + \beta^j K_j)\right]_\alpha{}^\beta\right\}^{-1} = \\ &= \left\{\exp\left[\mathrm{i}(\alpha^i J_i - \beta^j K_j)\right]^{\dot{\beta}}{}_{\dot{\alpha}}\right\}^{*-1T} \equiv \left\{(\bar{M}^{-1T})^{\dot{\beta}}{}_{\dot{\alpha}}\right\}^{*-1T}, \end{aligned} \tag{76}$$

regardless if in K_i we choose the plus or the minus sign. Therefore, the matrix $(\bar{M}^{-1T})^{\dot{\beta}}{}_{\dot{\alpha}}$ (which acts on $\bar{\phi}^{\dot{\alpha}}$) differs from the matrix $M_\alpha{}^\beta$ by taking the adjoint of the generators J_i, K_i, or, equivalently, by $J_i, -K_i$. This also shows that the the spinors ψ_α and $\bar{\phi}^{\dot{\alpha}}$ behave the same under rotations but inversely under boosts.

5.2 The Dirac equation

The momentum p^μ transforms as a vector, therefore it can be written as the bispinor $p_{\alpha\dot{\alpha}} = p^\mu \sigma_{\mu\alpha\dot{\alpha}}$. Analogously, introduce $\bar{p}^{\dot{\alpha}\alpha} = p^\mu \bar{\sigma}_\mu{}^{\dot{\alpha}\alpha}$. One might interpret those bispinors as the momentum operator and its adjoint acting on Weyl spinors. Then, the simplest equations that follow are

$$p_{\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = p^\mu \sigma_{\mu\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = 0 \quad \text{and} \quad \bar{p}^{\dot{\alpha}\alpha}\psi_\alpha = p^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha}\psi_\alpha = 0. \tag{77}$$

It is easy to show that $p\bar{p} = \bar{p}p = p^\mu p_\mu = p^2$. Hence, the above equations describe massless particles ($p^2 = 0$). The canonical substitution[1] $p_\mu \to -\mathrm{i}\partial_\mu$ yields the *Weyl equations*:

$$\boxed{\mathrm{i}\partial^\mu \sigma_{\mu\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = 0 \quad \text{and} \quad \mathrm{i}\partial^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha}\psi_\alpha = 0.} \tag{78}$$

A straightforward generalization of the above equations is given by

$$\boxed{\mathrm{i}\partial^\mu \sigma_{\mu\alpha\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = m\psi_\alpha \quad \text{and} \quad \mathrm{i}\partial^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha}\psi_\alpha = m\bar{\phi}^{\dot{\alpha}},} \tag{79}$$

where on the left hand side of both the above equations we had to use the conjugated spinor in order to match the indices. These equations correspond to the massive Dirac equation which makes the Weyl spinors dependent on each other. Note that it follows from (79) that both the Weyl spinors satisfy the *Klein-Gordon equation*:

$$(\Box + m^2)\psi_\alpha = 0 \quad \text{and} \quad (\Box + m^2)\bar{\phi}^{\dot{\alpha}} = 0. \tag{80}$$

5.3 The Dirac adjoint

Note that $S_{\mu\nu}$ is not hermitian. However, observe that $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ and hence $S_{\mu\nu}$ satisfies $S_{\mu\nu}^\dagger = \gamma_0 S_{\mu\nu}\gamma_0$, a condition called *pseudohermicity*. By further introducing the *Dirac adjoint* $\bar{\Psi} := \Psi^\dagger\gamma_0$, one can construct an inner product between Dirac spinors:

$$\boxed{\langle \Phi, \Psi\rangle_D := \bar{\Phi}\Psi = \Phi^\dagger\gamma_0\Psi.} \tag{81}$$

The above inner product is defined such that it is invariant under Lorentz transformations:

$$\begin{aligned} \langle \Phi, \Psi\rangle_D \to \langle \Phi', \Psi'\rangle_D &= \Phi^\dagger\gamma_0^2 \exp[\frac{\mathrm{i}}{2}\omega^{\mu\nu}S_{\mu\nu}]^\dagger \gamma_0 \exp[\frac{\mathrm{i}}{2}\omega^{\mu\nu}S_{\mu\nu}]\Psi = \\ &= \Phi^\dagger\gamma_0 \exp[-\frac{\mathrm{i}}{2}\omega^{\mu\nu}S_{\mu\nu}] \exp[\frac{\mathrm{i}}{2}\omega^{\mu\nu}S_{\mu\nu}]\Psi = \Phi^\dagger\gamma_0\Psi = \langle \Phi, \Psi\rangle_D. \end{aligned} \tag{82}$$

[1]We set $\hbar = c = 1$.