

# Exercises on General Relativity TVI TMP-TC1

Problem set 14, due 05.02 - 09.02

## Exercise 1 – Friedmann equations

- (i) Use previous results and derive the first and second Friedmann equations, namely in the usual coordinates the 00 and  $ij$  components of the Einstein field equations with a cosmological constant term:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu} \quad (1)$$

where the energy momentum tensor  $T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu}$  with the velocity  $u_\mu$ , energy density  $\epsilon$  and pressure  $p$ .

- (ii) Additionally, from  $\nabla_\mu T^{\mu 0} = 0$  derive the energy conservation equation.

## Exercise 2 – Cosmological models

Consider the Friedmann equation for  $\Lambda = 0$  and a flat universe filled with perfect fluid characterized by the equation of state  $\omega = \frac{p}{\epsilon}$ .

- (i) How does  $a(t)$  scale in terms of  $t$  for dust ( $\omega = 0$ ), radiation ( $\omega = 1/3$ ) and the cosmological constant ( $\omega = -1$ ). How do the energy densities scale in terms of  $a$ ?
- (ii) For which equation of state is  $\ddot{a} = 0$ ,  $\ddot{a} < 0$  and  $\ddot{a} > 0$ ? What is the corresponding physical situation in these cases?
- (iii) Find the evolution of  $a(t)$  for a flat universe with dust and cosmological constant using the results of i).
- (iv) Rewrite the Friedmann equations in conformal time, defined by

$$\eta(t) = \int_0^t \frac{dt'}{a(t')}. \quad (2)$$

In the following consider the three cases of an open, flat and closed universe.

- (v) Find the scale factor in terms of the conformal time for a dust dominated universe. What is the range for  $\eta$  for a closed dust dominated universe?

## Exercise 3 – Cosmological Redshift

The expansion of the Universe leads to a redshift of photon wavelength.

- (i) The cosmological redshift can be interpreted as doppler shift due to the relative motion of galaxies, with Hubble expansion. Using the Hubble law, estimate the difference in frequency  $\Delta\omega$  of light detected by two galaxies separated by the distance  $\Delta l$  and show that the frequency scales as  $1/a$ . *Hint: Choose a local inertial frame and argue why the derivation holds in general curved spacetimes*
- (ii) Redshift also plays a role in the detection of the velocity of massive particles. Let  $v_1$  be the velocity of a particle as measured by observer 1 and  $v_2$  the velocity measured by observer 2. Instead of using the argument above we will work in curved space immediately; show that the geodesic equation for the particles can be written as:

$$\frac{du_a}{d\lambda} - \frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} u^b u^c = 0 \quad (3)$$

Infer that the peculiar velocity scales as  $1/a$ .

- (iii) The so-called redshift parameter  $z$  is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{a(0)}{a(t_{\text{em}})} - 1 \quad (4)$$

Show that  $z$  can be used as an alternative measure of time, by deriving an integral expression for  $t(z)$ .

## General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:

Monday at 16:00 - 18:00 in B 138

There are six tutorials:

Monday at 12:00 - 14:00 in A 249

Thursday at 16:00 - 18:00 in A 449

Friday at 14:00 - 16:00 in C 113 and A 249

Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

[www.physik.uni-muenchen.de/lehre/vorlesungen/wise\\_17\\_18/tvi\\_tc1\\_gr/index.html](http://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html)