Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 14, due 05.02 - 09.02

Exercise 1 – Friedmann equations

(i) Use previous results and derive the first and second Friedmann equations, namely in the usual coordinates the 00 and ij components of the Einstein field equations with a cosmological constant term:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu} \tag{1}$$

where the energy momentum tensor $T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p\, g_{\mu\nu}$ with the velocity u_μ, energy density ϵ and pressure p.

(ii) Additionally, from $\nabla_\mu T^{\mu 0} = 0$ derive the energy conservation equation.

Exercise 2 – Cosmological models

Consider the Friedmann equation for $\Lambda = 0$ and a flat universe filled with perfect fluid characterized by the equation of state $\omega = \frac{p}{\epsilon}$.

(i) How does $a(t)$ scale in terms of t for dust ($\omega = 0$), radiation ($\omega = 1/3$) and the cosmological constant ($\omega = -1$). How do the energy densities scale in terms of a?

(ii) For which equation of state is $\ddot{a} = 0$, $\ddot{a} < 0$ and $\ddot{a} > 0$? What is the corresponding physical situation in these cases?

(iii) Find the evolution of $a(t)$ for a flat universe with dust and cosmological constant using the results of i).

(iv) Rewrite the Friedmann equations in conformal time, defined by

$$\eta(t) = \int_0^t \frac{\mathrm{d}t'}{a(t')} \,. \tag{2}$$

In the following consider the three cases of an open, flat and closed universe.

(v) Find the scale factor in terms of the conformal time for a dust dominated universe. What is the range for η for a closed dust dominated universe?

Exercise 3 – Cosmological Redshift

The expansion of the Universe leads to a redshift of photon wavelength.

(i) The cosmological redshift can be interpreted as doppler shift due to the relative motion of galaxies, with Hubble expansion. Using the Hubble law, estimate the difference in frequency $\Delta\omega$ of light detected by two galaxies seperated by the distance Δl and show that the frequency scales as $1/a$. *Hint: Choose a local inertial frame and argue why the derivation holds in general curved spacetimes*

(ii) Redshift also plays a role in the detection of the velocity of massive particles. Let v_1 be the velocity of a particle as measured by observer 1 and v_2 the velocity measured by observer 2. Instead of using the argument above we will work in curved space immediately; show that the geodesic equation for the particles can be written as:

$$\frac{du_a}{d\lambda} - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a} u^b u^c = 0 \tag{3}$$

Infer that the peculiar velocity scales as 1/a.

(iii) The so-called redshift parameter z is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{a(0)}{a(t_{\text{em}})} - 1 \tag{4}$$

Show that z can be used as an alternative measure of time, by deriving an integral expression for $t(z)$.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`