Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 7 extra, due December 8th

This problem sheet is discussed instead of the lecture.

Exercise 1 – Small exercises

(i) Compute the mass dimension of the scalar field ϕ and constants λ_n of the following action in eight space-time dimensions:

$$S = \int \mathrm{d}^8 x \left(-\tfrac{1}{2}\phi\Box\phi - \sum_{n=2}^{4} \frac{\lambda_n}{n!}\phi^n \right) .$$

(ii) Write the following tensors in a local coordinate basis using the basis theorem: (a) metric, (b) inverse metric, and (c) rank-(2,2) tensor T.

(iii) Calculate the 1-forms $\mathrm{d}h$ and $\mathrm{d}k$, where

$$h(x,y,z) = x^2y^2 + x^3z, \quad k(x,y) = \frac{1}{\sqrt{x^2+y^2}} .$$

(iv) Solve for the unknown vector X^α in the following equation:

$$\epsilon_{\alpha\beta\gamma}X^\beta A^\gamma = B_\alpha \quad \text{and} \quad X^\alpha C_\alpha = k, \quad \text{where} \quad A^\alpha B_\alpha = 0 \qquad (\text{in } \mathbb{R}^3) \tag{1}$$

where k is a non-zero scalar, A, B and C are linearly independent vectors and ϵ is the 3D totally antisymmetric tensor.

Exercise 2 – Working on a sphere

(i) Reconsider the stereographic projections of an n-sphere of problem set 7 for $n = 2$. The coordinates on $\mathbb{S}^2$ we want to use here is given by the stereographic pojection from the north pole $N := (0,0,1)$:

$$y_1 = \frac{x_1}{1-z}, \quad y_2 = \frac{x_2}{1-z} \tag{2}$$

Let the vector fields X and Y on $U^+ := \mathbb{S}^2 \setminus \{N\}$ be defined in these coordinates by

$$X = y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}, \quad Y = y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} . \tag{3}$$

Express these two vector fields in coordinates corresponding to the stereographic projection from the south pole $S := (0,0,-1)$.

(ii) Consider a surface $S \subseteq \mathbb{R}^3$ parametrized by a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^3, f(u^1,u^2)$. Let $g(x,y) = \sqrt{R^2 - x^2 - y^2}$ with a fixed real positive number R . The graph of f can be parametrised as follows:

$$f = \left(u^1, u^2, g(u^1,u^2)\right) . \tag{4}$$

This provides a parametrisation of the (open) northern hemisphere.
Judge whether this is a regular parametrisation of the surface S, i.e. $\frac{\partial f}{\partial u^1}$ and $\frac{\partial f}{\partial u^2}$ are linear independent.