Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 9, due January 7th

Exercise 1 – Examples of 1-forms and tensors

(i) Calculate the 1-forms $\mathrm{d}h$ and $\mathrm{d}k$, where

$$h(x, y, z) = 4x^2y + x^3z, \quad k(x, y) = \sqrt{x^2 + y^2}\ . \tag{1}$$

(ii) In 3 dimensional Euclidean space, decide whether the following maps are tensors:

$$T : (v, w) \mapsto 2v \times w - v(n \cdot w) \tag{2}$$
$$S : (v, w) \mapsto 2v \times w - (v \cdot v)(n \cdot w) \tag{3}$$
$$R : (v, w) \mapsto n \cdot (v \times w) - (n \cdot v)(n \cdot w) \tag{4}$$

with $v, w, n \in \mathrm{Vec}(\mathbb{R}^3)$ and n is fixed.
For each K for which it is possible determine the components $K^a{}_{bc}$ in a given basis.

Exercise 2 – Induced metrics on surfaces

Consider again the 2-sphere $N = \mathbb{S}^2$ embedded in $M = \mathbb{R}^3$. The coordinates on N and M are denoted as $x^\mu = (\theta, \phi)$ and $y^\mu = (x, y, z)$ respectively for $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. Then the map $\Phi : N \to M$ is given by

$$\Phi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\ . \tag{5}$$

Find the metric on $\mathbb{S}^2$ by pulling back the Euclidean metric from $\mathbb{R}^3$.
Now take the vector field defined on the surface N as

$$V = \cos\phi\,\partial_\phi + \sin\phi\,\partial_\theta \tag{6}$$

and check whether it is a unit vector and compute it's components in M.

Now consider another submanifold of $\mathbb{R}^3$, namely the surface parametrically given by

$$x = \sin^2\theta\cos\phi \tag{7}$$
$$y = \sin^2\theta\sin\phi \tag{8}$$
$$z = \cos\theta\sin\theta\ . \tag{9}$$

Find the induced metric for this surface. Which body is associated to this surface?

Exercise 3 – Surfaces of cylinders and cones

Consider the surface of a cylinder and a cone parameterized as

$$r_{\mathrm{cyl}}(h,\phi) : x = R\cos\phi, \quad y = R\sin\phi, \quad z = h \tag{10}$$

$$r_{\mathrm{cone}}(h,\phi) : x = ah\cos\phi, \quad y = ah\sin\phi, \quad z = h\ , \tag{11}$$

respectively with $\theta \in [0,\pi)$, $\phi \in [0, 2\pi)$ and constants $R, a \in \mathbb{R}^+ \setminus \{0\}$. Calculate the induced metrics of these surfaces embedded in $\mathbb{R}^3$. Are these surfaces locally/globally isometric to the Euclidean plane?

Exercise 4 – Levi-Civita connection

Consider $M = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ endowed with the metric

$$g_{ab} = \frac{1}{y^2}\delta_{ab}\ , \tag{12}$$

where δ is the Eucledian metric in $\mathbb{R}^2$. Compute the Levi-Civita connection of (M, g) by explicitly using the Koszul formula.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`