Chapter 1

Phase Transitions and the Renormalization group

1.1 Basic Problem

As the external conditions of a macroscopic system are changed the properties of the system can sometimes change dramatically. A good example of such a phenomenon is provided by a ferromagnet. When the temperature of a ferromagnet is increased above a certain temperature, called the *Curie temperature*, then the ferromagnet loses its magnetism and changes into a paramagnet. Furthermore in the neighborhood of the Curie temperature T_C the *susceptibility* χ diverges. We recall that if a ferromagnet is placed in an external magnetic field B its magnetization M changes. If B is now changed to $B + \delta B$ then M changes to $M + \delta M$. The susceptibility χ is defined as $\chi = \left(\frac{\partial M}{\partial B}\right)$. The manner in which the magnetization of a ferromagnet approaches zero as the temperature T of the system is increased to the Curie temperature of the system can be studied experimentally . It is found that in the absence of an external magnetic field and for T close to T_C

$$M = M_0 \left| \frac{T - T_C}{T_C} \right|^{\beta}$$

with $\beta \approx 0.33$ for many different ferromagnets. Similarly it has been found that $\chi(T)$ near $T \sim T_C$ behaves as

$$\chi(T) \propto \frac{1}{|T - T_C|^{\gamma}}$$

with $\gamma \approx 1.25$. Such behavior constitutes a *second order phase transition*. The transition is from the ferromagnetic phase at $T < T_c$, characterized by $M \neq 0$ for $B = 0$ to the paramagnetic phase at $T > T_c$ where $M = 0$ for $B = 0$.

Let us now give a concrete definition of a phase transition. A given equilibrium state of a macroscopic system can be described by an *order parameter* field. For a ferromagnet the order parameter field is the magnetization density. The order parameter field can be regarded as a mapping from the system (with coordinate

x) to an order parameter space. In a model where the magnetization density is given by a scalar function $M_z(\mathbf{x})$ this is the space of real numbers $\Re$. In general the order parameter space can be more complicated and the order parameter field need not be a scalar function.

Definition 1.1 A phase transition corresponds to the order parameter field changing qualitatively together with the emergence of singular behavior in the system.

For instance the order parameter field in the case of a ferromagnet is non-zero in the ferromagnetic phase, is zero in the paramagnetic phase and the susceptibility of the system diverges at the phase transition temperature. Determining a suitable order parameter field to characterize a phase is part of the task of a theory of phase transitions. If the order parameter field changes continuously from one phase to another, as in the case of a ferromagnet, the transition is said to be a continuous or second order phase transition. If it is discontinuous the transition is said to be *first order*. An example of a first order transition is when a solid melts to a liquid. The density of the system, which can be taken as the order parameter, changes discontinuously. A phase transition is a striking example of an emergent phenomenon. Starting off with only short range interactions between its microscopic magnetic moments, the system realizes long range correlations below T_C. We will now give an argument, based on a simple model of a ferromagnet, to show that it is impossible to understand the singular behavior of the susceptibility as a function of temperature. We will then give a second argument based on a simple model for the susceptibility χ to conclude that it is easy to understand the physical origin of the singular behavior of χ. Reconciling these two points of view will lead us to understand that certain infinite limits are important in statistical mechanics and to the *renormalization group* approach. On the way we will discuss an approach to phase transitions due to Landau.

We start with a model for a ferromagnet. We regard a ferromagnetic solid as being made out of a finite number of elementary magnets placed at locations throughout the solid. Due to quantum effects we simplify our model by assuming that each of these elementary magnets m can either point up ($m = +1$) or down ($m = -1$). Finally each elementary magnet interacts only with its nearest neighbor. A Hamiltonian for this model could be

$$\mathsf{H} = -g \sum_{n,i} m_i m_{i+n} - B \sum_i m_i.$$

where the first sum is over i as well as the nearest neighbors of i. Note H decreases if m_i, m_{i+n} have the same sign for $g > 0$. There are altogether a large but finite number of magnets in a ferromagnet. We are now ready to prove our theorem.

Theorem 1.1 In the model of a ferromagnet proposed the susceptibility cannot diverge.

Proof. In any statistical mechanics the partition function of a system once calculated determines the thermal properties of the system. For a magnetic system the

macroscopic variables are the magnetization of the system M, the external magnetic field B and the temperature T of the system. The canonical ensemble for this system is defined by the partition function

$$Z = \sum_{\{c\}} e^{-\beta H}, \qquad \beta = \frac{1}{kT}$$

where $\{c\}$ denotes the set of all configurations of the individual magnetic moments. We assume $\beta \neq \infty$ and define the free energy as usual by $F = -\frac{1}{\beta} \ln Z$. The susceptibility χ of the system is defined as in chapter **??**. We now show that χ cannot be a singular function of temperature. The proof is straight forward. Since our model involves a finite number of elementary magnets each of which interacts with a finite number of its neighbors and can exist in only two states, it follows that the number of configurations which have to be summed over to determine the partition function Z is finite. Each term of the sum is an analytic function of temperature and B. Since a finite sum of analytic functions is again an analytic function Z is an analytic function of the temperature and B. Furthermore each term $e^{-\beta H}$ is strictly positive. So Z is also strictly positive. Thus $F = -\frac{1}{\beta} \ln Z$ is an analytic function of the temperature and B. An analytic function of the temperature and the external magnetic field will continue to be analytic in those variables no matter how many times the function is differentiated. That is the definition of an analytic function. Therefore $\chi = -\frac{\partial^2 F}{\partial B^2}$ is a non-singular function of temperature, which concludes the proof. □

One response to this theorem might be to suggest that the theorem fails if we allow the number of elementary magnets to tend to infinity. This is because an infinite sum of analytic functions need not be analytic. An elementary example of this is the series $1 + x + x^2 + \ldots + x^N$. If we let $N \to \infty$ then the series tends to $\frac{1}{1-x}$ which has a singularity at $x = 1$. In order to analyse this possibility we will need to consider the statistical mechanics partition function in the limit in which the number of configurations is infinite. It is only in this limit that the phase transitions might be understood from this point of view.

Now for our second approach. This time we suppose that the external magnetic field B is changed to $B + \delta B(\mathbf{x})$, *i.e.* the change δB is position dependent. We expect that a change at $\mathbf{x}, \delta B(\mathbf{x})$, will produce a change in the magnetization δM not just at the point $\mathbf{x}$ but at other points as well. Indeed we might expect

$$\delta M(\mathbf{y}) \propto C_T(|\mathbf{x} - \mathbf{y}|)\, \delta B(\mathbf{x})$$

where $C_T(|\mathbf{x} - \mathbf{y}|)$ is a 'correlation-function' which determines the effect at $\mathbf{y}$ on the magnetization due to a change in the external field δB at $\mathbf{x}$. The total change $\delta M(\mathbf{y})$ is then expected to be

$$\delta M(\mathbf{y}) = \int d^3\mathbf{x}\, C_T(|\mathbf{x} - \mathbf{y}|)\, \delta B(\mathbf{x}).$$

We have assumed that the correlation function depends only on temperature and on the distance between the points $\mathbf{x}$ and $\mathbf{y}$. This is our simple model. The formula we have just guessed can be derived as a mathematical result within the framework of *linear response theory* and is known as *Kubo's formula*. For our own purpose the equation provides a convenient starting point for understanding the physical origin of the divergence of the susceptibility χ. Let us now suppose that δB is independent of $\mathbf{x}$ and let us set $\mathbf{y} = 0$. Then we have

$$\chi(0) = \frac{\delta M(0)}{\delta B} = \int d^3x \, C_T(|\mathbf{x}|)$$

If we suppose

$$C_T(|\mathbf{x}|) = \begin{cases} \alpha, & \text{for } |\mathbf{x}| \leq a(T), \\ 0, & \text{for } |\mathbf{x}| > a(T), \end{cases}$$

that is, a disturbance only propagates a distance $a(T)$, then

$$\chi(0) = \frac{4\pi\alpha}{3} a^3(T).$$

Thus $\chi(0)$ will diverge if $a(T)$ diverges, that is, if correlations in the system become infinite. From this point of view the divergence in this susceptibility is due to the fact that near a phase transition disturbances propagate over large distances, $a(T) \to \infty$. Our task is to reconcile the two approaches described. This will lead us to what is known as the renormalization group approach to phase transitions.

Let us start by verifying if by allowing the number of configurations to go to infinity we can indeed recover ferromagnetism in our model with the associated singular behaviour. To check this we consider a situation where the partition function can be exactly determined. The model we consider is the one dimensional Ising model discussed in chapter **??**. We recall that the model was defined by a Hamiltonian

$$H = -g \sum_{i=1}^{N} S_i S_{i+1} - B \sum_{i=1}^{N} S_i$$

In our calculation in chapter **??** we found that in the $N \to \infty$ limit the free energy F for this system is given by

$$F = -\frac{N}{\beta} \ln \left(e^{\beta g} \left[\cosh \beta B + \sqrt{\cosh^2(\beta B) - 2e^{-2\beta g} \sinh(2\beta g)} \right] \right)$$

The magnetization M and the susceptibility χ are then given by

$$M = -\frac{\partial F}{\partial B}, \quad \chi = -\frac{\partial^2 F}{\partial B^2}$$

Using this expression for F it is clear that $M = 0$ if $B = 0$ for $T = 0$ and χ is not singular. Thus our simple model fails to provide a model for a ferromagnet even in the $N \to \infty$ limit, *i.e.* a system for which $M \neq 0$, if $B = 0$, for some T. Our hope of getting a system to become ferromagnetic for $T < T_C$ with singular behavior for χ in the limit where the number of configurations was allowed to go to infinity has failed for this model. What went wrong? We now give an argument due to Peierls which explains why the model does not lead to ferromagnetism and why if the model is extended to two dimensions it should work!

1.2 Peirls Argument

The idea of Peierls was to start with an ordered phase of the system, that is, a phase in which all the spins point in the same direction, say, $S = +1$, even in the absence of an external magnetic field, and then to introduce a length L of 'disordered' (*i.e.* $S = -1$) spins. Peierls considered the change of the free energy F of the system for different temperatures when this was done. If F increases it means that the original ordered phase was stable so that the system could exist in a ferromagnetic phase while if F decreases it means the ordered phase was thermodynamically unstable. Indeed in Chapter **??** we established that the free energy F is minimal at equilibrium at constant temperature and under conditions when no work is done. We have by definition

$$F = U - TS$$

Keeping T fixed if the system is changed in the way we described we have

$$\Delta F = \Delta U - T\Delta S$$

Now observe that the change in U can be regarded as due to the change in the energy of the system. Setting $B = 0$ we note that if all $S_i = +1$ or all $S_i = -1$, the Hamiltonian has the same value. Thus if a disordered element of $S_i = -1$ of length L is introduced in a one-dimensional system with $(N - L)$ spins $S_i = +1$ then the energy of the system only changes at the boundaries. This is because it was assumed that we only have nearest neighbors interact in our model. For a one dimensional system there are only two boundaries. Thus

$$\Delta U = 2W,$$

where W is the energy change in replacing $S_i = 1, S_{i+1} = 1$ by $S_i = 1, S_{i+1} = -1$, *i.e.* $W = 2g$.

Next we have to calculate ΔS. To do this we recall that the entropy is given by $k \ln N_0$ where N_0 is the number of ways the configuration could be constructed. In the case of the one dimensional model N_0 is given by the number of ways in which a disordered length L can be introduced into the system. It is clear that

the disordered length can be made to start at $i = 1, 2, \ldots, N - L$. Thus N_0, the number of ways the disordered configuration can be introduced is equal to $N - L$. Therefore

$$\Delta F = 2W - kT \ln(N - L).$$

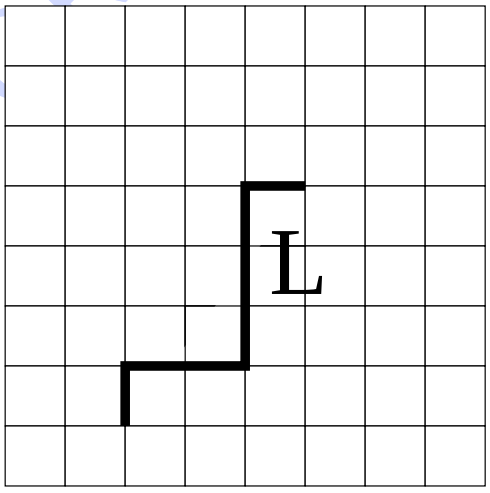
We are to take the $N \to \infty$ limit in order to allow a phase transition to occur as we saw. Thus $\Delta F < 0$ when the disordered element is introduced. Hence no ordered phase should be expected for the model. This is a reassuring result. It tells us that on general grounds a one dimensional model with nearest neighbor interactions cannot undergo a phase transition and this explains our failure to get a phase transition for the one dimensional Ising model.

Let us consider a generalization of the Ising model to two dimensions and apply Peierls' ideas. We will see that a phase transition is now possible and the argument gives a value for the critical (or Curie) temperature for the system. Our discussion is not intended to be complete but only to explain clearly the essential difference between the one and two dimensional models.

The model we now consider is given by

$$H = -g \sum_{i,j} \left(S_{i,j} S_{i+1,j} + S_{i,j} S_{i,j+1} \right) .$$

where the spins S_i are placed on a $N \times N$ periodic lattice. We start with an ordered phase of the system when all the spins $S_i = +1$. We again introduce a line of length L of disordered elements. A picture might be helpful.



In this case $\Delta U = (2WL)$ as introducing a line L disordered element in two dimensions introduces a boundary of length L as well. Next we have to determine the number of ways, N_0, such a line can be introduced in the system. We note that

- The line can start at any one of the points on the $N \times N$ lattice.

- From the starting point the next disordered element can be chosen in three ways (for the square lattice system we consider)

Thus

$$N_0 = N^2 3^L.$$

This simple way of counting does not take into account the fact that the defect line has no orientation and we are thus over counting by a factor of 2. This is, however, not relevant for our purpose. For concreteness let us count only configurations which are almost maximally disordered. For such configurations we have

$$L = fN^2$$

where f is a number between 0 and 1. Thus

$$\begin{aligned}\Delta F &= 2WfN^2 - kT \ln\left(N^2 3^{fN^2}\right) \\ &= 2WfN^2 - kT\left(fN^2 \ln 3 + \ln N^2\right).\end{aligned}$$

For large N the last term can be neglected so that

$$\Delta F \simeq fN^2 \left(2W - kT \ln 3\right).$$

Thus $\Delta F > 0$ if $2W > kT \ln 3$ or $kT < \left(\frac{2W}{\ln 3}\right)$. In the two dimensional case even with a nearest neighbor model the argument of Peierls suggests that a phase transition is allowed provided the temperature T of the system is less than $\frac{2W}{k \ln 3}$. This expectation is justified for two reasons. First, the numerical study of the two dimensional Ising model in chapter **??** confirms our expectations. Second, the two dimensional Ising model was solved exactly by Onsager and a phase transition with singular behavior for its susceptibility is found in the limit where the number of configurations is allowed to go to infinity.

Now that we know that statistical mechanics can indeed be used to study phase transitions we turn to a simple phenomenological approach for understanding phase transitions due to Landau.

1.3 Landau Theory of Phase Transitions

We have already made use of the fact that for any change of a system in which the temperature is kept fixed and no work is done by the system, the change of free energy, ΔF is always negative so that a state of equilibrium must be a minimum of F. Landau utilized this property of the free energy in his theory of phase transitions. Let us examine this approach for the case of a ferromagnet. The basic idea is to make a model for the free energy F near the Curie temperature T_C when the

system is still a ferromagnet. We know that for $T < T_C$ long range correlations are present, that is, the spin at lattice site $\mathbf{x}$ must point in the same direction as that at site $\mathbf{y}$ even when $\mathbf{x}$ and $\mathbf{y}$ are not adjacent. Otherwise the observed macroscopic magnetic properties of the system would not exist. The basic assumption underlying Landau's theory is that, near the critical temperature T_C, the properties of a ferromagnet can be described in terms of a magnetization density function $\mathbf{M}(\mathbf{x})$. The function $\mathbf{M}(\mathbf{x})$ can be defined by considering a volume element ΔV, large compared to the lattice cell volume, but small compared to the volume of correlated spins centred around the point $\mathbf{x}$. The magnetization of the volume element ΔV is defined to be $\mathbf{M}(\mathbf{x})\Delta V$. For this definition of $\mathbf{M}(\mathbf{x})$ to be useful it is important that $\mathbf{M}(\mathbf{x})$ should not be a rapidly varying function of position. Near the Curie temperature T_C we also expect $\mathbf{M}(\mathbf{x})$ to be small in amplitude. Since we are implicitly assuming in this approach that a spin-spin type of interaction is responsible for the phenomenon of ferromagnetism it seems reasonable to expect the free energy density to be a function of $\mathbf{M}(\mathbf{x}) \cdot \mathbf{M}(\mathbf{x})$. On the basis of arguments of this kind Landau proposed to introduce a functional $F_L[T, \mathbf{B}, \mathbf{M}]$ of the magnetization density $\mathbf{M}(\mathbf{x})$, temperature T and external magnetic field $\mathbf{B}(\mathbf{x})$ of the form

$$F_L[T, \mathbf{B}, \mathbf{M}] = F_L[T, \mathbf{B}, \mathbf{M} = 0] + \int d^3x \Big[a(T)\, \mathbf{M}(\mathbf{x}) \cdot \mathbf{M}(\mathbf{x}) + b(T) \left(\mathbf{M}(\mathbf{x}) \cdot \mathbf{M}(\mathbf{x})\right)^2 + \ldots + c(T) \sum_{i,j} \left(\nabla_j M_i(\mathbf{x})\right) \cdot \left(\nabla_j M_i(\mathbf{x})\right) + \ldots - \mathbf{B} \cdot \mathbf{M}(\mathbf{x}) \Big],$$

The free energy $F_L(T, \mathbf{B})$ is then obtained by minimizing $F_L[T, \mathbf{B}, \mathbf{M}]$ with respect to $\mathbf{M}$. Note that the temperature dependent coefficients $a(T), b(T), c(T), \ldots$ are assumed to be smooth functions of temperature. We will simplify the model function by assuming $\mathbf{B}(\mathbf{x})$ acts along the z-direction and that $\mathbf{M}(\mathbf{x})$ only has components in the z-direction. Then we have

$$F_L[T, B_z, M_z] = F_L[T, B_z, M_z = 0] + \int d^3x \left[a(T)\, M_z^2(\mathbf{x}) + b(T)\, M_z^4(\mathbf{x}) + \ldots + c(T) \left(\nabla M_z(\mathbf{x})\right) \cdot \left(\nabla M_z(\mathbf{x})\right) + \ldots - B_z(\mathbf{x})\, M_z(\mathbf{x}) \right].$$

The expression for the Landau free energy F_L is expected to be useful when T is close to the Curie temperature T_C. In this region $M_z(\mathbf{x})$ is expected to be small and we also expect $(\nabla M_z(\mathbf{x}) \cdot \nabla M_z(\mathbf{x}))$ to be small. Because of these reasons we will from now on ignore the effect of the higher powers of $M_z(\mathbf{x})$ and higher gradient terms. To determine the equilibrium configuration of the magnetization $M_z(\mathbf{x})$ we have to minimize the free energy with respect to $M_z(\mathbf{x})$. Using

$$\delta F_L = \int d^3x [2a(T)\, M_z(\mathbf{x}) + 4b(T)\, M_z^3(\mathbf{x}) - 2c(T)\, \nabla^2 M_z(\mathbf{x}) - B_z(\mathbf{x})] \delta M_z(\mathbf{x})$$

we see that vanishing of δF for arbitrary $\delta M_z(\mathbf{x})$ requires

$$2a(T)\, M_z(\mathbf{x}) + 4b(T)\, M_z^3(\mathbf{x}) - 2\left(c(T)\, \nabla^2 M_z(\mathbf{x})\right) = B_z(\mathbf{x}).$$

Suppose now that $B_z(\mathbf{x})$ does not depend on $\mathbf{x}$ and let us see if a solution for $M_z(\mathbf{x})$ independent of $\mathbf{x}$ is possible. Such an $\mathbf{x}$ independent solution must satisfy

$$[2a(T) M_z + 4b(T) M_z^3] = B_z.$$

Now we ask if it is possible to construct a solution with the property that $M_z \neq 0$ when $B_z = 0$ and $T < T_C$. Setting $B_z = 0$ we find three solutions

$$M_z = 0$$

$$M_z = \pm\sqrt{\frac{-a(T)}{2b(T)}}.$$

We would also like the solution to have the property

$$M_z = 0, \quad \text{when} \quad T > T_C$$
$$M_z \neq 0, \quad \text{when} \quad T < T_C.$$

If such a solution is possible then the expression for F represents a model for a ferromagnet. As we have stressed this model is constructed to represent a ferromagnet near its Curie temperature. We also assume that the coefficient functions $a(T), b(T), c(T)$ are all smooth functions of temperature. We thus expect the M_z^4 term to be small compared to the M_z^2 term. It is then reasonable to replace $b(T)$ by $b(T_C) = b_0$, a constant. Finally setting $a(T) \simeq a_0(T - T_C)$ we have as our equilibrium $\mathbf{x}$ independent solution

$$M_z = 0 \qquad \text{or}$$

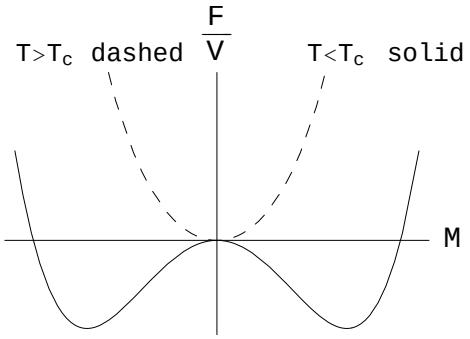
$$M_z = \pm\sqrt{\frac{-a_0(T - T_C)}{2b_0}}.$$

Let us now discuss the Landau free energy for arbitrary $M_z(\mathbf{x})$ independent of $\mathbf{x}$ and $a(T) = a_0(T - T_C)$, $b(T) = b$, $B_z = 0$ respectively. Ignoring the M_z independent contribution in F_L, we find

$$F_L(T, M_z) = V[a_0(T - T_C) M_z^2 + b_0 M_z^4\ ].$$

Let us now discuss the choice of the constants a_0 and b_0 in turn. We distinguish three cases, depending on the sign of the coefficients a_0 and b_0. In each case we plot $\frac{F_L}{V}$ as a function of M_z.

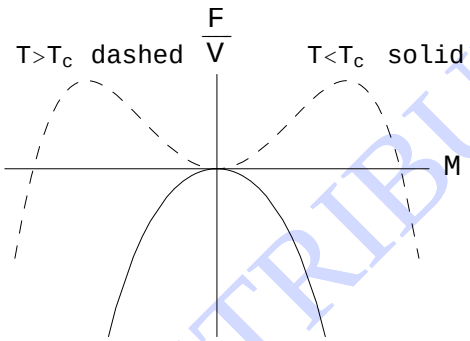
1. $a_0 > 0, b_0 > 0$:



$$M_1 = -\sqrt{\frac{a_0\,(T_c - T)}{b_0}} \qquad M_2 = +\sqrt{\frac{a_0\,(T_c - T)}{b_0}}$$

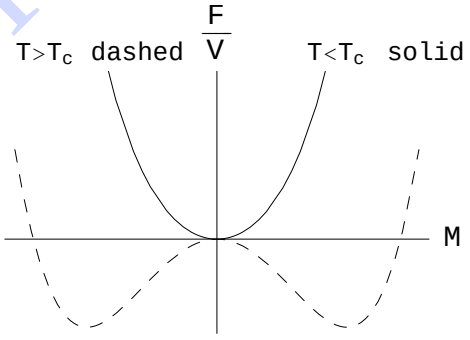
Note that M_1 and M_2 have lower free energy than the solution $M_z = 0$. Thus according to Landau's theory the system would settle to M_1 or M_2 as its equilibrium magnetization configuration. Observe also that for $T > T_C$ the free energy $\frac{F_L}{V}$ is a convex function of M_z. Thus $M_z = 0$ is the minimum of the free energy for $T > T_C$.

2. $a_0 > 0, b_0 < 0$:



In this case the equilibrium distribution $M_z = 0$ unstable for any temperature. The same situation arises for $a_0 < 0, b_0 < 0$ which simply amounts to interchanging the two graphs for $T < T_C$ and $T > T_C$.

3. $a_0 < 0, b_0 > 0$:



This case is not physically reasonable since it leads to spontaneous magnetization above the critical temperature!

To summarize, if Landau's expression for the free energy is to represent a ferromagnet we must choose $a_0 > 0, b_0 > 0$. The alert reader will have noticed that the free energy plotted in the above figures is not a convex function. This appears to be in contradiction with our result in chapter **??** that the free energy F is a thermodynamic potential and as such a convex function of its extensive variables. This puzzle is resolved by noting that the true free energy is, in fact, the convex

hull of the functions plotted above. This is because the equilibrium states can be mixtures of pure phases, rather than just pure phases which we have implicitly assumed by setting $M_z = constant$ independent of the position. We consider a concrete example of mixing in problem 1.4.

Having determined the signs of a_0 and b_0 we now want to explore the predictions of the Landau theory. From the equation

$$(2a_0 (T - T_C) + 4b_0 M_z^2) M_z = B_z$$

it then follows that for $B_z = 0$ and $T < T_C$

$$M_z = \pm \sqrt{\frac{a_0}{2b_0}} (T_c - T)^{\frac{1}{2}} \ .$$

Firthermore, for $T = T_C$,

$$M_z^3 = \frac{B_z}{4b_0} .$$

Finally we note that if B_z is changed to $B_z + \delta B_z$, the corresponding equilibrium distribution M_z can be written as $M_z + \delta M_z$. We thus get

$$2a_0 (T - T_c) \delta M_z + 12 b_0 M_z^2 \delta M_z = \delta B_z .$$

Setting $B_z = 0$, we find

$$\chi = \left(\frac{\delta M_z}{\delta B_z} \right)_{B_z = 0} = \begin{cases} \frac{1}{2a_0 (T - T_C)} & \text{for } T > T_C \\ \frac{1}{4a_0 (T_C - T)} & \text{for } T < T_C \ . \end{cases}$$

where we have substituted $M_z = 0$ for $T > T_C$ and $M_z = \sqrt{\frac{a_0}{2b_0}} (T_c - T)^{\frac{1}{2}}$ for $T < T_C$ respectvely. Comparing this with the critical behaviour stated at the beginning of this chapter we see that Landau's theory qualitatively reproduces the expected singular behaviour although the precise nature of the singularity is not reproduced exactly. We will come back to this point at the end of this section.

It is also possible to get a rather precise statement regarding long-range correlations within the framework of Landau's theory. To do this we recall our linear response relation introduced in section 1.1

$$\delta M_z (\mathbf{x}) = \int d^3 y \ C_T (|\mathbf{x} - \mathbf{y}|) \delta B_z (\mathbf{y})$$

where we regard δM_z as the change in the equilibrium magnetization density brought about by changing the external magnetic field by $\delta B_z (\mathbf{y})$. From the equations which describe the equilibrium magnetization density we have with $C(T) = c_0$,

$$2a (T) M_z (\mathbf{x}) + 4b_0 M_z^3 (\mathbf{x}) - 2c_0 \nabla^2 M_z (\mathbf{x}) = B_z (\mathbf{x})$$

and

$$2a\left(T\right)\left(M_z+\delta M_z\right)+4b_0\left(M_z+\delta M_z\right)^3-2c_0\nabla^2\left(M_z+B_z\right)=B_z+\delta B_z.$$

From these two equations it follows that

$$[2a_0\left(T-T_c\right)+12b_0M_z^2-2c_0\nabla^2]\delta M_z\left(\mathbf{x}\right)=\delta B_z\left(\mathbf{x}\right)\ .$$

Substituting the linear response relation expression for $\delta M_z\left(\mathbf{x}\right)$ we get

$$\left(2a_0\left(T-T_C\right)+12b_0M_z^2-2c_0\nabla^2\right)C_T\left(|\mathbf{x}-\mathbf{y}|\right)=\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)$$

where we have written $\delta B_z\left(\mathbf{x}\right)=\int d^3y\delta\left(\mathbf{x}-\mathbf{y}\right)\delta B\left(\mathbf{y}\right)$. Setting $B_z=0$ and rescaling the coordinates as $(\mathbf{u},\mathbf{v})=2c_0(\mathbf{x},\mathbf{y})$ we get

$$\left(2\bar{a}_0\left(T-T_C\right)-\nabla^2\right)C_T\left(|\mathbf{u}-\mathbf{v}|\right)=\delta^{(3)}\left(\mathbf{u}-\mathbf{v}\right)\ ,$$

where $\bar{a}_0=a_0/(2c_0)^3$. This result is valid in the region $T>T_C$. For $T<T_C$ we get an analogous equation with $\bar{a}_0\rightarrow-2\bar{a}_0$. The solution of this equation can be inferred from the following result.

Lemma 1.2 The differential equation

$$\left(m^2-\nabla^2\right)C_T\left(|\mathbf{u}|\right)=\delta^3\left(\mathbf{u}\right)$$

has the solution

$$C_T\left(|\mathbf{u}|\right)=\frac{1}{4\pi}\frac{e^{-m|\mathbf{u}|}}{|\mathbf{u}|}\ ,$$

modulo solutions of the homogenous equation.

Proof. To prove the lemma let us write the delta function in momentum space as

$$\delta^3\left(\mathbf{u}\right)=\int\frac{d^3k}{\left(2\pi\right)^3}\,e^{i\mathbf{k}\cdot\mathbf{u}}$$

and similarly

$$C_T\left(|\mathbf{u}|\right)=\int\frac{d^3k}{\left(2\pi\right)^3}\,e^{i\mathbf{k}\cdot\mathbf{u}}\,\Delta\left(|\mathbf{k}|\right).$$

Substituting these expressions in the differential equation we get

$$\left(m^2+|\mathbf{k}|^2\right)\Delta\left(|\mathbf{k}|\right)=1$$

Then, performing the inverse Fourier transform we end up with

$$C_T\left(|\mathbf{u}|\right)=\int\frac{d^3k}{\left(2\pi\right)^3}\,e^{i\mathbf{k}\cdot\mathbf{u}}\,\frac{1}{|k|^2+m^2}.$$

In order to evaluate this integral let us choose a coordinate system in which $\mathbf{u}$ points along the z-axis. Changing to polar coordinates we have

$$\begin{aligned} C_T(|\mathbf{u}|) &= \int_0^\infty \frac{dkk^2}{(2\pi)^3} \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos(\theta) e^{ik\cos\theta|\mathbf{u}|} \left(\frac{1}{k^2+m^2}\right) \\ &= \frac{1}{2\pi^2}\frac{1}{|\mathbf{u}|}\int_0^\infty \frac{dkk}{k^2+m^2}\sin k|\mathbf{u}| \\ &= \frac{1}{4\pi^2}\frac{1}{|\mathbf{u}|}\mathrm{Im}\left(\int_{-\infty}^{\infty}\frac{dkk}{k^2+m^2}e^{ik|\mathbf{u}|}\right) \end{aligned}$$

This integral can be evaluated by closing the contour of integration above and below the real line respectively leading to

$$C_T(|\mathbf{u}|) = \frac{1}{4\pi}\frac{e^{-m|\mathbf{u}|}}{|\mathbf{u}|}$$

which is the claimed solution. □

To continue we restore the original coordinates $(\mathbf{x}, \mathbf{y})$ and introduce the *correlation length* $\xi^2 = c_0/a(T)$. Then

$$C_T(|\mathbf{x}-\mathbf{y}|) = \frac{1}{4\pi}\frac{e^{-\frac{|\mathbf{x}-\mathbf{y}|}{\xi}}}{|\mathbf{x}-\mathbf{y}|}\,.$$

The corresponding result for $T < T_C$ is obtained simply by replacing $a(T) \to -2a(T)$ in the correlation length. We notice that as $\xi \to \infty$ as $T \to T_C$. Thus Landau's theory is in qualitative agreement with the intuitive idea, introduced earlier in section 1.1 that long-range correlations are generated in a ferromagnet as $T \to T_C$. Another point to note is that if δB_z were $\mathbf{x}$ independent, then as we saw before

$$\begin{aligned} \chi(0) &= \frac{\delta M_z(0)}{\delta B_z} \\ &= \int d^3y\, C_T(|\mathbf{y}|)\,. \end{aligned}$$

From the expression for $C_T(|\mathbf{y}|)$ it then follows that

$$\chi \sim \xi^2 \to \infty \quad \text{as} \quad T \to T_C.$$

Let us summarize the results obtained from Landau's approach. The approach focused on long-range correlations and suggested that the singular behavior of the susceptibility was due to such correlations when $T \to T_C$. The approach also predicts that the relation between different macroscopic parameters involves power

laws.

$$\begin{aligned} M_z &\sim (T_C - T)^{\beta}\,, \qquad && T \to T_C \\ M_z &\sim B_z^{\frac{1}{\delta}}, && T = T_C \\ \chi &\sim \frac{1}{(T_C - T)^{\gamma}}, && T \to T_C. \end{aligned}$$

with $\beta = \frac{1}{2}$, $\gamma = 1$ and $\delta = 3$. The parameters β, δ, γ are called *critical exponents* and are measured experimentally. The experimental values for these parameters $\beta \simeq 0.33, \delta \simeq 4.5$, and $\gamma \simeq 1.2$ are found for different ferromagnets with different lattice structures and widely differing values for the Curie temperature T_C. These parameters thus are a *universal* property of the ferromagnetic phase transition. This is also a feature of Landau's theory. Thus Landau's theory is in qualitative agreement with experiment.

1.4 Renormalization Group

Although Landau's theory is in good qualitative agreement with experiment there is room for improvement on the quantitative level concerning the critical exponents. This will lead us to the renormalization group. Our treatment will be rather brief as our aim is to explain the basic ideas involved in the renormalization group approach and not explore technical issues. The reader is referred to some of the books listed at the end of the chapter for a more thorough treatment of this important topic.

The renormalization group can be approached in two different ways. There is the real space approach and there is the field theory approach. In both approaches the crucial physical input is the assumption that a certain length scale of the system approaches infinity near a phase transition.

Let us start by looking at the real space approach. Instead of considering a model for the free energy we now consider the partition function directly. Since the dynamics of a physical system is completely determined by the Hamiltonian we can think of the partition function as a function of the Hamiltonian H and the temperature. Thus

$$\begin{aligned} Z &= Z\,(\beta, \mathsf{H}) \\ &= Z\,(\mathsf{H})\,, \;\; \beta = \frac{1}{kT} \;\; \text{fixed}\,. \end{aligned}$$

If we now assume that there is a natural length scale ξ, the correlation length, which should be used as the unit of length and that $\xi \to \infty$ as $T \to T_C$ then Z better be *scale invariant*, since in this limit there is no physical length left to set the scale. To illustrate this we may consider a model for a ferromagnet in which elementary magnets are present in a cubic lattice of size a_0. Scale invariance then means that the physics at scales of the order ξ should not change if the microscopic

a_0 is replaced by $a_L = La_0$ as long as $La_0 \ll \xi$. If the Hamiltonian H represents the dynamics of the system when the scale of the system is a_0 and H_L represents the dynamics when the scale is a_L then the statement that the physical properties of the system are scale invariant means in particular that

$$Z(\mathsf{H}) = Z(\mathsf{H}_L) .$$

This is the key assumption of the real space renormalization group approach. Now, since the correlation length ξ depends solely on the Hamiltonian of the system we can write $\xi = \xi(\mathsf{H})$. On the other hand, since ξ is a length, dimensional analysis implies

$$\xi(\mathsf{H}_L) = \frac{1}{L}\xi(\mathsf{H}) \ .$$

The intuitive picture behind this equation is simply that the same correlation length is measured in different units namely a_0 and $a_L = La_0$. As for the free energy *density* for the system (which we will denote gain by F) it is given by

$$F = -\frac{1}{\beta V}\ln Z(\mathsf{H}) \ ,$$

and similarly

$$F(\mathsf{H}_L) = -\frac{1}{\beta V_L}\ln Z(\mathsf{H}_L) .$$

We can repeat the argument we used for the correlation length. Since the volume V is fixed and only the length scale is changed we must have $V_L = L^{-3}V$ and therefore

$$F(\mathsf{H}_L) = L^D F(\mathsf{H})$$

for a D-dimensional system. Of course, these equations acquire content only if we know how H_L and H are related or, in mathematical terms, if the transformation function τ

$$\tau(\mathsf{H}) = \mathsf{H}_L \ ,$$

which relates H and H_L is known. The function τ is the *real space renormalization group transformation*. We will now sketch how in principle τ can be obtained given a certain Hamiltonian and how once τ is known it is possible to calculate the actual critical exponents. We shall find that our qualitative argument leads to the statement that the free energy is a *generalized homogeneous function*. We will explain what this means and point out that this has experimentally testable implications. Our aim is to make the general strategy of the renormalization group approach clear. We will not go into the details of how the critical exponents are

actually determined in this approach. We will learn how to calculate the critical exponents when we discuss the field theory approach to the renormalization group. For concreteness let us consider a two dimensional ferromagnetic system described in terms of an Ising model. Let

$$\begin{aligned}\tilde{\mathsf{H}} &\equiv \beta \mathsf{H} \\ &= g\left(T\right) \sum_{n,i} S_i S_{i+n} - B\left(T\right) \sum_i S_i\end{aligned}$$

where $g(T) = \beta g_0$ and $h(T) = \beta B_0$ are the "coupling constants" and $S_n = \pm 1$. The summation in the first term is over nearest neighbors. When the length scale of the system is a_0, the nearest neighbors of a given spin variable S_n are located at a distance a_0 from S_n. If the length scale is changed from a_0 to La_0 the nearest neighbors of S_n^L will be at a distance La_0. This assumes, of course, that H and H_L have the same structure. In H, S_n took the value ± 1. For H_L we have to determine the corresponding range of values of S_n^L.

Since the total number of spin variables N is fixed by changing length scales from a_0 to La_0 the number of degrees of freedom the system are reduced. We can then interpret S_n^L as the value of the block of spins surrounding S_n and contained within a cube of side La_0. For $T \sim T_c$ if $\xi \gg La_0$ we might expect all the spins within the block to be correlated, that is, if one of the spins within the block takes on the value $+1$ then so do all the other spins in the block. If there are N_L spins in the block then S_n^L can take values $\pm N_L$. The picture leads us to expect that H_L differs from H in three ways. First the length scale a_L is different from a_0. Second, the magnitude of the spin variables differs and finally the coupling constants differs. Thus

$$\tilde{\mathsf{H}}_L = g_L\left(T\right) \sum_{n,i} S_n^L S_{n+i}^L - B_L\left(T\right) \sum_i S_i^L$$

The renormalization group transformation τ in this case corresponds to the set of equations :

$$g_L = U_L\left(g, B\right)$$

$$B_L = V_L\left(g, B\right)$$

Clearly this process can be repeated; we can start with H_L and generate H_{L^2} , from H_{L^2} generate H_{L^3} etc. All of these Hamiltonians form a class of "equivalent" Hamiltonians. In particular, they lead to the same partition function. Note for this argument to work it is necessary for the correlation length ξ to be such that $\xi >> L^{N_0}$ $(N_0 = 1, 2, 3, \ldots)$. It is possible that the sequence $\mathsf{H}, \mathsf{H}_L, \mathsf{H}_{L^2}, \ldots$ converges to a H^* such that

$$\tau\left(\mathsf{H}^*\right) = \mathsf{H}^*$$

The Hamiltonian H^* is then a *fixed point* of the renormalization group transformation τ. Note that the existence of a fixed point is a property of τ and hence will not depend greatly on the starting Hamiltonian H. Of course, there could be several fixed points so that one class of Hamiltonian which acted on by τ will converge to H_1^* , another class which converges to some other fixed point H_2^* and so on.

Let us study qualitatively the implication of assuming that H^* describes the behavior of a physical system near the critical temperature. To keep things simple we will first assume a vanishing external field, $B = 0$. The renormalization group transformation of g_L is then

$$g_L = U_L\,(g)$$

We want to investigate the effect of repeated applications of the transformation τ. For this we consider

$$g_{L'L} = U_{L'}\,(g_L)$$

We assume that $U_{L'}\,(g_L)$ is a smooth function of L' and g_L. We then have

$$\frac{dg_L}{dL} = \frac{1}{L}\lim_{\delta\to 0}\,\frac{g_{(1+\delta)L} - g_L}{\Delta L} \equiv \frac{1}{L}u(g_L)\ .$$

The characteristic property of H^* is $\tau(\mathsf{H}^*) = \mathsf{H}^*$. In terms of g_L this implies $g^* = U_{L'}(g^*)$ or

$$L\frac{dg^*}{dL} = 0$$

i.e. $u(g^*) = 0$. Furthermore, since we assume $u(g_L)$ is a smooth function of g_L we have, for g_L close to g^*

$$u(g_L) = (g_L - g^*)y\,, \qquad y = \left(\frac{\partial u_L}{\partial g_L}\right)_{g_L = g^*}\ .$$

Near the critical point we therefore have

$$L\frac{dg_L}{dL} = (g_L - g^*)y\ .$$

What we have just done is known as the *linearisation* of the renormalization group equation near a critical point. Note that the right hand side of this equation does not explicitly depend on L.

Next we study the correlation length ξ. We have seen already that that

$$L\xi\,(g_L) = \xi(g)$$

Since relation is true for all L we have in particular, that $(L + \Delta L)\,\xi\,(g_{L+\Delta L}) = \xi\,(g) = L\xi\,(g_L)$ or, letting $\Delta L \to 0$,

$$\xi\,(g_L) + \left(L\frac{dg_L}{dL}\right)\frac{\partial \xi}{\partial g_L} = 0$$

Substituting the expression for g_L close to g^* we then end up with

$$\xi\left(g_L\right) + \left(g_L - g^*\right) y \frac{\partial \xi}{\partial g_L} = 0 \; .$$

The solution to this differential equation is easily found to be

$$\xi\left(g_L\right) \sim \left(g_L - g^*\right)^{-\frac{1}{y}}$$

Since we identify scale invariance with the system being at critical temperature we identify $g^* = \frac{g_0^*}{kT_c}$ and similarly, $g_L = \frac{g}{kT}$. To continue it is convenient to introduce the dimensionless variable

$$t = \left(\frac{T_c - T}{T_c}\right)$$

Substitution into the equation for ξ then gives $\xi(t) \sim t^{-\frac{1}{y}}$. Thus, $\xi \to \infty$, provided $y > 0$ as $t \to 0$ with critical exponent $\gamma = \frac{2}{y}$.

We now want to study the scaling properties of the free energy $F\left(t_L, B_L\right)$, near the critical temperature. Here, $t_L = \left(g_L - g^*\right)$ measures the deviation away from the fixed point. From the renormalization group equation for g_L we then have $t_L = L^y t$. As for the external field B_L it is not hard to see that the linearisation of the renormalization group equation near the fixed point leads to the scaling behaviour

$$B_L = L^x B \; , \qquad \text{with} \quad x = \left(\frac{\partial V_L(g_L, B_L)}{\partial B_L}\right)_{g^*, B^*}$$

Combining this with the scaling behaviour of the free energy, $F\left(\mathsf{H}_L\right) = L^D F\left(\mathsf{H}\right)$, we get

$$F\left(t_L, B_L\right) = L^D F\left(t, B\right) \; ,$$

or,

$$F\left(L^y t, L^x B\right) = L^D F\left(t, B\right)$$

If we define $\lambda = L^D$, so that $L = \lambda^{\frac{1}{D}}$ we have

$$F\left(\lambda^{\frac{y}{D}} t, \lambda^{\frac{x}{D}} B\right) = \lambda F\left(t, B\right)$$

This is the definition of a *generalized homogeneous function*. If the function $U_{L'}\left(g_L, B_L\right)$ and $V_{L'}\left(g_L, B_L\right)$ are known, then the coefficients x and y can be calculated. These would determine the coefficients $a_t = \frac{y}{D}, a_B = \frac{x}{D}$. Hence all the critical properties of the system could be calculated as we now demonstrate. We have

Theorem 1.3 If $F(\lambda^{a_t}t, \lambda^{a_B}B) = \lambda F(t, B)$ with λ, a constant and a_t, a_B two parameters then:

$$\begin{aligned} M(t,0) &= (t)^{\frac{1-a_B}{a_t}} M(1,0) \\ M(0,B) &= B^{\frac{1-a_B}{a_B}} M(0,1) \\ \chi &= t^{\frac{1-2a_B}{a_t}} \chi(1,0) \end{aligned}$$

that is, we have the following expressions for the critical exponents $\beta = \frac{1-a_B}{a_t}$, $\delta = \frac{a_B}{1-a_B}$ and $\gamma = \frac{2a_B-1}{a_t}$.

Proof. We start with the thermodynamic identities:

$$M(t,B) = -\left(\frac{\partial F}{\partial B}\right)_t \quad \text{and} \quad \chi(t,B) = \left(\frac{\partial M}{\partial B}\right)_t$$

Using the generalized homogeneous function structure of the free energy we have

$$\begin{aligned} M(\lambda^{a_t}t, \lambda^{a_B}B) &= \frac{\partial F(\lambda^{a_t}t, \lambda^{a_B}B)}{\partial(\lambda^{a_B}B)} \\ &= \lambda^{-a_B}\frac{\partial F(\lambda^{a_t}t, \lambda^{a_B}B)}{\partial B} \\ &= \lambda^{-a_B}\frac{\partial(\lambda F(t,B))}{\partial B} \\ &= \lambda^{1-a_B}M(t,B) \end{aligned}$$

Thus

$$M(\lambda^{a_t}t, \lambda^{a_B}B) = \lambda^{1-a_B}M(t,B)$$

If we set $B = 0$ and $\lambda^{a_t} \cdot t = 1$ we obtain the first identity in the theorem:

$$M(t,0) = t^{\frac{1-a_B}{a_t}} M(1,0) \ .$$

In order to prove the second identity we set $t = 0$ and $\lambda^{a_B}B = 1$ which gives

$$M(0,B) = B^{\frac{1-a_B}{a_B}} M(0,1)$$

Finally, using $\chi(t,B) = \frac{\partial M}{\partial B}$ leads to

$$\chi(\lambda^{a_t}t, \lambda^{a_B}B) = \lambda^{1-2a_B}\chi(t,B)$$

Setting $B = 0$ and $\lambda^{a_t}t = 1$ gives

$$\chi(t,0) = t^{\frac{1-2a_B}{a_\tau}} \chi(1,0)$$

which completes the proof. □

To summarise, we have three experimental parameters β, γ, δ expressed in terms of two theoretical parameters. a_B, a_t. Thus, an immediate corollary is the *scaling law* $\gamma = \beta(\delta - 1)$. Putting in numbers $\gamma = 1.25, \beta = 0.33, \delta = 4.5$, we see it is approximately correct. This scaling law is just one of several relations that can be obtained in this way. We leave the derivation of the other scaling laws as a guided problem at the end of this chapter.

1.5 Critical Exponent Calculations

We now turn to the problem of calculating the various critical exponents that appeared in the last section borrowing techniques developed in our exposé on quantum field theory. The method presented here will improve on Landau's theory while reducing to the latter in a certain limit. The idea is to represent Landau's model as a certain approximation to the partition function of some quantum field theory for the order parameter. In analogy with our treatment of the path integral approach to quantum field theory we would like to write the partition function Z as a functional integral. As a guide to our intuition we use Landau's ansatz for the free energy. We thus write

$$Z = \int D\Phi \; e^{-\frac{1}{\hbar} S[\Phi]}$$

where

$$S[\Phi(\mathbf{x})] = \int [\Phi(\mathbf{x})(-\nabla^2)\Phi(\mathbf{x}) + V(\Phi)] d^D x$$

is a functional of of the order parameter field with $V(\Phi)$ yet to be determined. As we will see below, this ansatz reproduces Landau's ansatz for the free energy in a saddle point approximation. A few comments are in order to justify these manipulations.

1. The functional integral above is *not* derived from a concrete Hamiltonian as we did in chapter **??**. The path integral should be understood rather as a *statistical averaging* over different configurations of the order parameter $\Phi(\mathbf{x})$.
2. We have redefined the order parameter field as $\sqrt{\beta\hbar}\Phi(\mathbf{x}) \to \Phi(\mathbf{x})$. The parameter $\hbar$ appearing in front of the action should thus not be interpreted as the fundamental Planck constant. Rather is is an auxiliary counting parameter introduced for convenience
3. If Φ is viewed as a fundamental field, then, as indicated in section **??**, the above expressions can be thought as the euclidan path integral representation of a relativistic quantum field theory with action $S[\Phi(x)]$.

Having made these comments we will now proceed by applying the usual perturbative techniques from quantum field theory to the above functional integral, that

is, we approximate $S[\Phi]$ by its functional Taylor expansion about $\Phi_0(\mathbf{x})$, which, in turn, is defined by the condition $\frac{\delta S}{\delta \Phi} = 0$. Thus

$$S[\Phi] = S[\Phi_0] + \frac{1}{2!}\int (\Phi - \Phi_0)_x (\Phi - \Phi_0)_y \frac{\delta^2 S}{\delta\Phi_x \delta\Phi_y} d^D x d^D y \,.$$

where we use the notation $(\Phi - \Phi_0)_x = \Phi(\mathbf{x}) - \Phi_0(\mathbf{x})$. We can think of the functional derivatives introduced as generalizing the familiar notion of partial derivatives to continuous variables. Let us explain: Recall if Φ was a n-tuple, namely $\mathbf{\Phi}$ with components Φ_i, $i = 1, \ldots, N$ and S was a function of Φ_i then a Taylor expansion of S for $\mathbf{\Phi}$ close to $\mathbf{\Phi}_0 = \Phi_{0i}$ would be

$$\begin{aligned} S(\mathbf{\Phi}) \;\; &\simeq \;\; S(\mathbf{\Phi}_0) + \sum_{i=1}^{N} (\Phi - \Phi_0)_i \left(\frac{\partial S}{\partial \Phi_i}\right)_{\mathbf{\Phi}_0} \\ &\quad + \frac{1}{2!}\sum_{i=1}^{N}\sum_{j=1}^{N} (\Phi - \Phi_0)_i)(\Phi - \Phi_0)_j \left(\frac{\partial^2 S}{\partial\Phi_i \partial\Phi_j}\right)_{\mathbf{\Phi}_0} + \ldots \end{aligned}$$

If we differentiate the components Φ_j with respect to Φ_i the result is simply

$$\frac{\partial \Phi_j}{\partial \Phi_i} = \delta_{ij} \,.$$

We now formally extend these ideas to $\Phi(\mathbf{x})$, where we think of $\Phi(\mathbf{x})$ as the components Φ_x of an "infinite dimensional vector" Φ. Then the expansion of $S[\Phi(\mathbf{x})]$ in a functional Taylor series about $\Phi_0(\mathbf{x})$ should have the structure,

$$\begin{aligned} S[\Phi(\mathbf{x})] \;\; &\simeq \;\; S[\Phi_0(\mathbf{x})] + \int_x (\Phi - \Phi_0)_x \left(\frac{\delta S}{\delta \Phi_x}\right)_{\Phi_0(\mathbf{x})} \\ &\quad + \frac{1}{2!}\int_x (\Phi - \Phi_0)_x (\Phi - \Phi_0)_y \left(\frac{\delta^2 S}{\delta\Phi_x \delta\Phi_y}\right)_{\Phi_0(\mathbf{x})} + \ldots \end{aligned}$$

where $\int d^D x$ replaces $\sum_i$ by analogy with the discrete case. Similarly, the basic rule of "functional differentiation" should be

$$\frac{\delta\Phi(\mathbf{x})}{\delta\Phi(y)} = \delta^D(\mathbf{x} - \mathbf{y})$$

that is the D-dimensional Dirac delta function replaces the Kronecker delta δ_{ij} for the discrete case.

We recall that a stationary point (actually a function) of $S[\Phi]$ corresponds to finding a solution to the classical equation,

$$\delta S = 0.$$

which corresponds to solving the Euler-Lagrange equations.

$$\frac{\partial L}{\partial \Phi_0} = \partial_i \left(\frac{\partial L}{\partial(\partial_i \Phi_0)}\right)$$

Such a $\Phi_0(\mathbf{x})$ is the classical solution. A classical approximation would correspond to replacing $S[\Phi]$ by $S[\Phi_0]$ in Z. A semi-classical approximation would involve replacing $S[\Phi]$ by

$$S[\Phi_0] + \frac{1}{2!}\int d^D x \int d^D y (\Phi - \Phi_0)_x (\Phi - \Phi_0)_y \left(\frac{\delta^2 S}{\delta\Phi_x \delta\Phi_y}\right)_{\Phi_0}$$

First let us consider the classical approximation to the partition function Z. Approximating the functional integral by its saddle point we find

$$Z \simeq e^{-\frac{1}{\hbar}S[\Phi_0]}$$

On the other hand we have in the canonical ensemble

$$Z = e^{-\beta F}\,.$$

Thus we can identify the free energy of statistical mechanics with $S[\Phi_0]$ of our field theory

$$F = \frac{1}{\beta\hbar} S(\Phi_0)\,.$$

It is now clear how to generalize Landau's theory. Instead of the saddle point, or classical approximation we consider the full path integral

$$Z(\beta, g, m) = \int D\Phi\; e^{-\frac{1}{\hbar}S[\Phi]}$$

with

$$\begin{aligned} S[\Phi] &= \int d^D x [\frac{1}{2}\Phi(\mathbf{x})(-\nabla^2)\Phi(\mathbf{x}) - V(\Phi)] \\ V(\Phi) &= \frac{1}{2}m^2\Phi^2(\mathbf{x}) - \frac{g}{4!}\Phi^4(\mathbf{x}) \end{aligned}$$

that is, we take for $S[\Phi]$ Landau's free energy F_L, replacing $M_z(\mathbf{x})$ by $\Phi(\mathbf{x})$. From our discussion of the path integral approach to quantum field theory, we know that a perturbation expansion for the partition function and correlation functions in the coupling constant g can be constructed. Our aim is then to see if by choosing special values for the coupling constants m^2 and g the partition function Z can be made scale invariant. The system then will be close to a phase transition configuration. Using the Landau form for S(Φ) we will analyse the corresponding field theory and see if for special values of the coupling constants the partition function Z can be made scale invariant. We would like to study the above functional integral within the framework of quantum field theory. To carry out this program a few results about functional integrals are needed which we will present in the next section.

1.6 Correlation Functions

In section 1.1 we have emphasized the relevance of correlation functions for explaining the the singular behavior of the susceptibility near the critical temperature, T_C. The following result shows how the two point correlation functions are related to the path integral representation of the partition function. To begin with we extend the result **??** for finite dimensional integrals formally to the case of a field $\Phi(\mathbf{x})$ as

Result 1.4

$$\int D\Phi\, e^{-\frac{1}{2}(\Phi,\mathsf{A}\Phi)+(J,\Phi)} = (\det \mathsf{A})^{-\frac{1}{2}}\, e^{\frac{1}{2}(J,\mathsf{A}^{-1}J)}$$

where

$$(\Phi,\mathsf{A}\Phi) = \int d^D x\, \Phi(x)(-\nabla^2 + m^2)\Phi(\mathbf{x})$$
$$(J,\Phi) = \int d^D x\, J(\mathbf{x})\Phi(\mathbf{x})$$

The determinant of the operator $\mathsf{A} = (-\nabla^2 + m^2)$ has to be suitably defined in terms of the non-zero eigenvalues of the differential operator $(-\nabla^2 + m^2)$.

This is an important identity since it allows us to express the correlation functions of an arbitrary number of fields in terms of derivatives of the partition sum with respect to the "external field" $J(\mathbf{x})$. We will show this explicitly for the case of the two point function.

Result 1.5 The two point correlation function

$$C_T(\mid \mathbf{x} \mid) = \frac{\int D\Phi\, e^{-(\Phi,\mathsf{A}\Phi)}\Phi(\mathbf{x})\Phi(0)}{\int D\Phi\, e^{-(\Phi,\mathsf{A}\Phi)}}\,, \qquad \mathsf{A} = (-\nabla^2 + m^2)$$

satisfies the partial differential equation

$$(-\nabla_x^2 + m^2)C_T(\mid \mathbf{x} \mid) = \delta^3(\mathbf{x})$$

This shows, in particular that $C_T(\mid \mathbf{x} \mid)$ agrees with the correlation function introduced in section 1.3 up to a solution of the homogenous equation $(-\nabla_x^2 + m^2)f(\mathbf{x}) = 0$.

Proof. In order to verify this claim we consider

$$\int D\Phi\, e^{-\frac{1}{2}(\Phi,\mathsf{A}\Phi)}\Phi(x)\Phi(y)$$

Using result 1.4 this becomes

$$(\det \mathsf{A})^{-\frac{1}{2}} \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} e^{\frac{1}{2}(J,\mathsf{A}^{-1}J)} \mid_{J=0}$$

where

$$(J, \mathsf{A}^{-1}J) = \int d^D x \int d^D y J(\mathbf{x}) \mathsf{A}^{-1}(\mathbf{x}, \mathbf{y}) J(\mathbf{y})$$

After expanding the exponential we will have to evaluate

$$\frac{\delta}{\delta J(\mathbf{x})} \frac{\delta}{\delta J(\mathbf{y})} [1 + \frac{1}{2}(J, \mathsf{A}^{-1}J) + \frac{1}{2!}\frac{1}{4}(J, \mathsf{A}^{-1}J)^2 + \dots] \mid_{J=0}$$

To continue we recall the functional derivatives given in section 1.5

$$\frac{\delta J(\mathbf{z})}{\delta J(\mathbf{x})} = \delta^D(\mathbf{x} - \mathbf{z}) \,.$$

Since J is to be set equal to zero at the end, the only term which contributes to the result is

$$\frac{\delta}{\delta J(\mathbf{x})} \frac{\delta}{\delta J(\mathbf{y})} \frac{1}{2}(J, \mathsf{A}^{-1}J) \mid_{J=0}$$
$$= \frac{1}{2}[\mathsf{A}^{-1}(\mathbf{x}, \mathbf{y}) + \mathsf{A}^{-1}(\mathbf{y}, \mathbf{x})] \,.$$

Since $\mathsf{A}^{-1}(\mathbf{x}, \mathbf{y}) = \mathsf{A}^{-1}(\mathbf{y}, \mathbf{x})$ is a hermitean operators we then have

$$\frac{\int D\Phi e^{-(\Phi, \mathsf{A}\Phi)} \Phi(\mathbf{x}) \Phi(0)}{\int D\Phi e^{-(\Phi, \mathsf{A}\Phi)}} = \mathsf{A}^{-1}(\mathbf{x}, \mathbf{y})$$

To conclude the proof we have to determine $\mathsf{A}^{-1}(\mathbf{x}, \mathbf{y})$ by solving the equation

$$\mathsf{A}\mathsf{A}^{-1} = \mathsf{I} \,, \qquad \text{or} \qquad \mathsf{A}\mathsf{A}^{-1}(\mathbf{x}, \mathbf{y}) = \delta^D(\mathbf{x}, \mathbf{y}) \,.$$

Upon substitution of $\mathsf{A} = (-\nabla^2 + m^2)$ the equation for $A^{-1}(\mathbf{x}, \mathbf{y})$ is given by

$$(-\nabla_x^2 + m^2) A^{-1}(\mathbf{x}, \mathbf{y}) = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \,,$$

which, in turn, establishes the result. □

1.7 Epsilon Expansion

Before we consider the corrections to the Landau approach let us recall how $C_T(\mid \mathbf{x} \mid)$ determines the singular behavior of the susceptibility χ. As explained in section 1.3 we have

$$\chi = \int d^3x \; C_T(\mid \mathbf{x} \mid) = (2\pi)^3 \Delta(0)$$

where $\Delta(\mathbf{k})$ is the Fourier transform of $C_T(\mid \mathbf{x} \mid)$. In particular if we ignore the interaction term, $\frac{g}{4!} \int d^D \Phi^4(x)$, we have just seen that $\Delta^{g=0}(0) = \frac{1}{m^2}$ and thus

$\chi = \frac{(2\pi)^3}{m^2}$. Setting $m^2 \propto (T_c - T)$ Landau's result is recovered. The singular behavior of χ is thus determined by the way $m^2 \to 0$ as $T \to T_C$.

The way to improve on Landau's result is now clear. We should study the full propagator

$$C_T^g(\mid \mathbf{x} \mid) = \frac{\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi) - \frac{g}{4!}\int d^D x \Phi^4} \Phi(\mathbf{x})\Phi(0)}{\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi) - \frac{g}{4!}\int d^D x \, \Phi^4}}$$

in the region where the parameters m and g of the theory are close to values which lead to a scale invariant partition function. The susceptibility χ is then determined by calculating $\Delta^g(0)$ corresponding to $\int C_T^g(\mid \mathbf{x} \mid)$.

Let us evaluate the lowest order modification to $C_T(\mid \mathbf{x} \mid)$ brought about by the interaction term $\frac{g}{4!}\int d^D x \Phi^4(\mathbf{x})$. To do this we expand the interaction in powers of g

$$e^{-\frac{g}{4!}\int \Phi^4(\mathbf{x}) d^D x} \simeq 1 - \frac{g}{4!}\int \Phi^4(\mathbf{x}) d^D x$$

Then to O(g) :

$$\begin{aligned} C_T(\mid \mathbf{x}_1 - \mathbf{x}_2 \mid) &= C_T^0(\mid \mathbf{x}_1 - \mathbf{x}_2 \mid) - \frac{g}{4!}\frac{\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi)} \Phi(\mathbf{x}_1)\Phi(\mathbf{x}_2) \int d^D y \, \Phi^4(\mathbf{y})}{\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi)}} \\ &+ \frac{g}{4!}\frac{\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi)} \Phi(\mathbf{x}_1)\Phi(\mathbf{x}_2) \int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi)} \int d^D y \, \Phi^4(\mathbf{y})}{\left(\int D\Phi \; e^{-(\Phi,\mathsf{A}\Phi)}\right)^2} \end{aligned}$$

The first term on the right hand side is just our result for the two point correlator in the absence of interactions. The second and the third term can now be evaluated in terms of C_T^0 using results 1.4 and 1.5 leading to

$$C_T(\mid \mathbf{x}_1 - \mathbf{x}_2 \mid) = C_T^0(\mid \mathbf{x}_1 \mid) - \frac{12g}{4!}\left(\int C_T^0\left(\mid \mathbf{x}_1 - \mathbf{y} \mid\right) C_T^0(0) C_T^0(\mid \mathbf{x}_2 - \mathbf{y} \mid) d^D y\right)$$

The factor 12 is a combinatorial factor arising when applying result 1.4. Just like in our discussion in quantum field theory in chapter **??** the above equation has a simple graphical interpretation. The right hand side can be represented by represented by drawing all *connected graphs* with two external lines corresponding to $\Phi(\mathbf{x}_1)$ and $\Phi(\mathbf{x}_2)$ and one vertex corresponding to $\int \Phi^4(\mathbf{x}) d^D x$, that is

The combinatorial factor 12 then simply counts the number of ways to connect the legs of the vertex with itself and the external lines, $\Phi(\mathbf{x}_1)$ and $\Phi(\mathbf{x}_2)$. Thus we recover the Feynman rules established in **??**. This is as it should be since as we have already mentioned in the remarks above we are treating the path integral like

a fictious quantum field theory with action $S[\Phi]$. The reason that we draw only connected Feynman graphs is due to the fact that the disconnected, or *vacuum graphs* are cancelled by the last term in the equation for C_T. Let us now rewrite our result in terms of the momentum space correlators $\Delta(\mathbf{k})$ and $\Delta^0(\mathbf{k})$. Defining

$$\begin{aligned} C_T(|\ \mathbf{x}\ |) &= \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}.\mathbf{x}} \Delta(\mathbf{k}) \\ C_T^0(|\ \mathbf{x}\ |) &= \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}.\mathbf{x}} \Delta^0(\mathbf{k}) \end{aligned}$$

we get

$$\begin{aligned} \Delta(\mathbf{k}) &= \Delta^0(\mathbf{k}) - 12\Delta^0(\mathbf{k})(\frac{g}{4!}\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2+m^2)})\Delta(\mathbf{k}) \\ &= \Delta^0(\mathbf{k}) - \frac{1}{2} g \Delta^0(\mathbf{k})\Sigma(0)\Delta^0(\mathbf{k}) \end{aligned}$$

where

$$\Sigma(0) = \int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2+m^2)}$$

In fact this integral is logarithmically divergent for $D = 3$. This is a common feature in quantum field theory which is addressed by adding suitable *counter terms* to the actions $S[\Phi]$. This process which is called *renormalization* is described extensively in the literature. For our purpose we can avoid this complication since we will only be interested in the m-dependent part of $\Sigma(0)$. Since we are interested in the susceptibility we may set $k = 0$. Then

$$\begin{aligned} \Delta(0) &\simeq \Delta^0(0) - \frac{g}{2}\Delta^0(0)\Sigma(0)\Delta^0(0) \\ &= \frac{1}{m^2} - \frac{g}{2m^4}\Sigma(0) \\ &\equiv \frac{1}{M^2} \end{aligned}$$

From our result relating the susceptibility χ to $\Delta(0)$ we have $\chi = \frac{(2\pi)^3}{M^2}$. Hence the singular behavior of χ can be determined, as a function of temperature, once we determine the way M depends on temperature. This in turn will follow from the way M depends on m since we know that $m^2 \propto T - T_c$. We shall look for a dependence of M on m of the form $M \propto m^{1+c(g)}$. This will change the singular behavior of χ from that predicted by Landau theory.

At this stage, the singular behavior of χ seems to depend on the value of the coupling constant g of the system. However, as we have emphasized at the end of section 1.5, in order for our quantum field theory model to describe a physical system near a phase transition, the theory must be scale invariant. Now since $m^2 \propto (T_C - T)$ scale invariance can be achieved at the critical temperature by

setting $g = 0$. In this case we recover the results from Landau theory. The question which we now want to address is whether taking higher order corrections into account we will find other non-trivial values of g leading to a scale invariant theory. For this we first introduce an *effective coupling*. This can be done by considering the four point function

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int D\phi e^{-(\phi, \mathsf{A}\phi) - \frac{g}{4!}\int \phi^4} \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)$$

Expanding the interaction term $e^{-\frac{g}{4!}\int \phi^4}$ we get

$$\int D\phi e^{-(\phi,\mathsf{A}\phi)}\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)[1 \quad - \quad \frac{g}{4!}\int \phi^4(x)d^D x + \quad (\frac{g}{4!})^2\left(\frac{1}{2!}\int \phi^4(x)d^D x\right)^2 + ...]$$

Now we consider the term linear in the coupling constant g, that is

$$\int D\phi e^{-(\phi,\mathsf{A}\phi)}\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)(\frac{-g}{4!})\int \phi^4(x)d^D x\ .$$

Using result 1.4 this gives

$$\frac{-g}{4!}\int C^0(\mathbf{x}_1 - \mathbf{x})C^0(\mathbf{x}_2 - \mathbf{x})C^0(\mathbf{x}_3 - \mathbf{x})C^0(\mathbf{x}_4 - \mathbf{x})d^D x$$

In momentum space this becomes

$$= \frac{-g}{4!}\int \prod_{i=1}^{4} d^D\mathbf{k}_i (\frac{1}{{\mathbf{k}_1}^2 + m^2})(\frac{1}{{\mathbf{k}_2}^2 + m^2})(\frac{1}{{\mathbf{k}_3}^2 + m^2})(\frac{1}{{\mathbf{k}_4}^2 + m^2})\delta^D(\sum_{i=1}^{4}\mathbf{k}_i)$$

where $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ represent the momenta of four particles. The coupling constant $\frac{-g}{4!}$ is obtained from this expression by picking the term from the integral with $\mathbf{k}_i = 0$ and multiplying this expression by $(m^2)^4$. This term represents the strength of the interaction between four zero-momentum particles. We will take this to be the definition of the coupling constant of the theory.

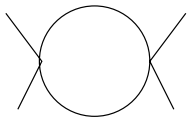
Let us now look at the term which is of second order in g. We have

$$\frac{1}{2}\int D\phi e^{-(\phi,\mathsf{A}\phi)}\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)(\frac{-g}{4!})^2(\int \phi^4(\mathbf{x})d^D x)^2$$

Using result 1.4 this can be easily evaluated. A typical term obtained by this procedure is

$$(\frac{-g}{4!})^2\frac{1}{2!}\int d^D x \int d^D y C^0(\mathbf{x}_1 - \mathbf{x})C^0(\mathbf{x}_2 - \mathbf{x})(C^0(\mathbf{x} - \mathbf{y}))^2 C^0(\mathbf{x}_3 - \mathbf{y})C^0(\mathbf{x}_4 - \mathbf{y})$$

and corresponds to the Feynman diagram



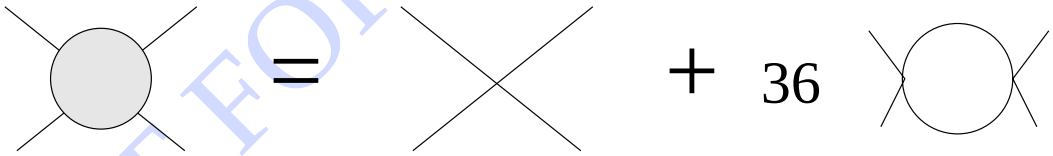
Writing this in momentum space, we get

$$= (\frac{-g}{4!})^2 \frac{1}{2!} \int \prod_{i=1}^{4} d^D\mathbf{k}_i (\frac{1}{{\mathbf{k}_1}^2 + m^2})(\frac{1}{{\mathbf{k}_2}^2 + m^2})(\frac{1}{{\mathbf{k}_3}^2 + m^2})(\frac{1}{{\mathbf{k}_4}^2 + m^2})$$
$$\times \delta^D(\sum_{i=1}^{4} \mathbf{k}_i) \left(\int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^2} \frac{1}{((\mathbf{k}_1+\mathbf{k}_2-q)^2+m^2)^2} \right)$$

Altogether there are $2({}^4C_2)^2$ such terms. They correspond to the number of terms generated by applying result 1.4 to evaluate the integral. From the graphical representation this factor can be understood as follows. There are $({}^4C_2)^2$ ways of selecting two lines from the two vertices and there are two ways of joining these to form the required graph. The contribution of these terms to the effective coupling constant can be obtained by again picking the term with $\mathbf{k}_i = 0$ from the integral and multiplying them by $(m^2)^4$. This gives

$$({}^4C_2)^2.2!(\frac{-g}{4!})^2 \frac{1}{2!} \left(\int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^2} \right)$$

If we ignore higher order terms we are lead to define the effective coupling g_R of the field theory in terms of the linear and quadric contribution to the four point function. Graphically we can write this as



Thus we have

$$\frac{1}{4!} g_R = \frac{1}{4!} g - (\frac{g}{4!})^2 \frac{1}{2!} ({}^4C_2)^2 2 \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^2}$$

i.e.

$$g_R = g - \frac{3}{2} g^2 \int \frac{d^Dq}{(2\pi)^D} \frac{1}{(q^2+m^2)^2}$$

Observe that g_R depends on m. Near a phase transition we would like g_R to be a scale invariant quantity and m to go to zero. This would lead to a scale invariant partition function.

In order to implement these ideas we start by first determining the scale dimensions of the different objects in the field theory. We observe that in units where the

Boltzman constant and $\hbar$ are unity $S[\Phi]$ must be a dimensionless quantity since otherwise $e^{-S[\Phi]}$ is ill-defined. By examining each term of $S[\Phi]$ we can establish the dimension "d" of the different quantities appearing in $S[\Phi]$. Introducing a length scale L, we find that

$$\begin{aligned} d_\Phi &= L^{\frac{2-D}{2}} \\ d_m &= L^{-1} \\ d_g &= L^{D-4} \end{aligned}$$

We recall that the parameter $m^2 \propto (T_c - T)$ and hence it is a variable. It is thus convenient to select m^2 as the length scale of our system. This allows us to study $\frac{\partial M^2}{\partial m^2}$ as a function of m^2 keeping g fixed. The parameter g is to be fixed so that the system is scale invariant. We have,

$$\begin{aligned} \frac{\partial M^2}{\partial m^2} &= 1 - \frac{g}{2}(m^2)^{(\frac{D-4}{2})}(\frac{A}{D-4}) \\ A &= \frac{2\Gamma(3-D/2)}{(2\sqrt{\pi})^D} \, , \end{aligned}$$

where we have used the result

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)^n} = \frac{1}{(2\sqrt{\pi})^D} m^{2(\frac{D}{2}-n)} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}$$

and

$$\Gamma(2-\frac{D}{2}) = 2\frac{\Gamma(3-\frac{D}{2})}{(D-4)}$$

to evaluate the integral

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)^2} \, .$$

Similarly we have for the effective coupling

$$g_R = g - \frac{3}{2} g^2 (m^2)^{\frac{D-4}{2}} \frac{A}{(D-4)} \, .$$

The method of evaluating integrals by treating the dimension D as a parameter is known as the *dimensional regularization method*. It is very useful for regulating potentially divergent integrals and is commonly used in quantum field theory. For problems in critical phenomena it is also convenient to regard $(4-D) = \epsilon$ as "small" and use it as an expansion parameter. This is known as the *epsilon expansion*. Writing

$$g_R = u_R m^{-2(D-4)/2} = x^{\epsilon/2} u_R$$

with $\epsilon = 4 - D$, and $x = m^2$, we have

$$x^{\epsilon/2} u_R = x^{\epsilon/2} u - \frac{3}{2} u^2 x^{\epsilon/2} \frac{A}{D-4}$$

We now implement our requirement that g_R^* be scale invariant that is independent of x, or $x\frac{\partial g_R^*}{\partial x} = 0$. This implies

$$\frac{\epsilon}{2}[u^* + \frac{3}{2}(u^*)^2 \frac{A}{\epsilon}] = 0$$

There are two solutions $u^* = 0, u^* = -\frac{2\epsilon}{3A}$. Let us now return to the expression for

$$\frac{\partial M^2}{\partial x} \mid_{g \text{ fixed}} = 1 - \frac{g}{2}(x)^{\frac{D-4}{2}}(\frac{A}{D-4})$$

We want to determine the dependence of M^2 on x due to interactions. If we write $M^2 = Bx^{1+c(g)}$ then

$$x\frac{d}{dx}\ln(\frac{\partial M^2}{\partial x}) = c(g)$$

In our case

$$\ln(\frac{\partial M^2}{\partial x}) = \ln(1 - \frac{g}{2}x^{\frac{D-4}{2}}(\frac{A}{D-4}))$$

So that

$$x\frac{d}{dx}\ln(\frac{\partial M^2}{\partial x}) = -A\frac{g}{4}x^{\frac{D-4}{2}} = -\frac{uA}{4}$$

Since $gx^{\frac{D-4}{2}} = u$, we have $c = -\frac{uA}{4}$. We found that there were two values for u which give rise to a scale invariant g_R^*. Namely $u^* = 0$. This gives c=0 and implies

$$\chi \sim \frac{1}{\mid T - T_c \mid}$$

which is Landau's result. The second solution is $u^* = -\frac{2\epsilon}{3A}$ with $\epsilon = 4 - D$. This gives $c = \frac{\epsilon}{6}$ and

$$\chi \sim \frac{1}{\mid T - T_c \mid^{1+\frac{\epsilon}{6}}}$$

In particular, for $D = 3$, we have $\epsilon = 1$ so that the critical exponent γ is changed from the Landau value of 1 to $(1 + \frac{1}{6}) \simeq 1.2$. Experimentally, for many systems, as we stated earlier $\gamma \simeq 1.25$. Thus choosing $u^* = -\frac{2\epsilon}{3a}$ improves on Landau's original result. In table 1.7 we compare the values for the critical exponents of different calculations with some experimental values. This gives an idea of the quality of the different approximate methods. Note that the predictions from

Exponent	ϵ-expansion	Landau	$2D$-Ising	$3D$-Ising	Experiment
α	$1/6$	0	0 (log. div.)	0.12	$0-0.14$
β	$1/3$	0.5	$\frac{1}{8}$	0.31	$0.32-0.39$
γ	$7/6$	1	1.75	1.25	$1.3-1.4$
δ	4	3	15	5	$4-5$
$\frac{(\beta\delta-\gamma)}{\beta}$	$1/2$	1	1	1	0.93 ± 0.08

Table 1.1 Critical exponents and scaling laws from ϵ-expansion (at $O(\epsilon)$), Landau theorie, exact results in the $2D$-Ising model, numerical results in the $3D$-Ising model and a variety of experimental values to illustrate the universal behavior. The exponnent α is defined in problem 1.1.

Landau theory improve as the dimensions of the system increases. This is due to fact that fluctuations which are important in low dimensions are neglected in the Landau theory.

Of course, $\epsilon = 1$ is not small and thus higher order corrections should be taken into account. Our aim in this section was to emphasise the physical ideas which underlie the real space and field theory approaches to critical phenomena and to give a flavour of the way a calculation of critical exponents is actually carried out. In order to do these calculations to greater accuracy a more elaborate machinery involving renormalizations is required. The specialised books listed at the end of the section maybe consulted for further details.

Problems

Problem 1.1 Consider the Landau model with

$$F_L(T,\Phi) = F_L(T,0) + \frac{r_0(T)}{2}\Phi^2 + u_0\Phi^4\,,$$

and let $F(T) \equiv F_L(T,\Phi)|_{\Phi=\langle\Phi(T)\rangle}$ be the free energy. Here $\langle\Phi(T)\rangle$ is determined by requiring that it minimizes $F_L(T,\Phi)$. We will assume that $r_0(T) = a_0\frac{T-T_c}{T_c}$ with $a_0 > 0$. In particular, $r_0(T)$ changes sign at $T = T_c$. Show that

1. S is continuous at $T = T_c$ and that
2. c_V is discontinuous. Compute the critical exponent α for c_V defined through $c_V \propto \left(\frac{T-T_c}{T_c}\right)^{\alpha}$. Hint: use that near T_c we can approximate $c_V \simeq T_c\frac{\partial S}{\partial T}$ (why?)

Problem 1.2 Discuss the consequences of a cubic term in the Landau model

$$F_L(T,\Phi) = F_L(T,0) + \frac{r_0(T)}{2}\Phi^2 + s_o\Phi^3 + u_0\Phi^4\,.$$

Sketch $F_L(T,\Phi)$ as a function of Φ for $T > T_c$ and $T < T_c$ and show that $\langle\Phi(T)\rangle$ is a discontinous function of T. What is the interpretation of this jump?

Problem 1.3 Revisit the Landau theory in section 1.3 from the second view point introduced in problem **??**. How will the corresponding formulas in section 1.3 be modified?

Problem 1.4 Convexity of the free energy: The free energy as a function of the order parameter plotted in section 1.3 is not convex. This is because in section 1.3 we considered only the pure phase where the order parameter is constant. In this problem we consider the possibility of a mixture of phases. For this we start with the Landau free energy for fixed $T_0 < T_C$

$$F_L\left[T_0,\Phi,V\right] = F_L(T_0,0,V) + \int d^3x \Big[\frac{1}{2}(\nabla\Phi)^2 - \frac{\tau}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\Big]\,,$$

where $\tau,\lambda > 0$, $\Phi(\mathbf{x})$ is the order parameter and $F_0(T_0,V)$ is the free energy for $\Phi = 0$. Assuming that $\Phi = \Phi_0$, a constant, the free energy as a function Φ_0 takes the form of the first plot in section 1.3 with two minima at $\Phi_0 = \pm\sqrt{\tau 3!/\lambda}$.

1. Show that if we allow Φ to depend on one coordinate, z say, then the free energy has another extremum for

$$\Phi_1(z) = \phi_0 \tanh\frac{z-z_0}{\xi}\,,$$

   where $\xi = \sqrt{2/\tau}$.
2. Show that for $-\frac{L}{2} \le z \le \frac{L}{2}$, with $L >> \xi$ the average value for the order parameter is given by

$$\langle\Phi\rangle = -\Phi_0\frac{2z_0}{L}\,.$$

   Thus $\langle\Phi\rangle$ as a function of z_0 interpolates between $-\Phi_0$ and Φ_0.
3. Show that for $L\to\infty$ the function $\frac{F_L[T_0,\Phi_1,V]}{V}$ is the convex hull of $\frac{F_L[T_0,\Phi_0,V]}{V}$ between $-\Phi_0$ and Φ_0.

In this concrete example we thus see explicitly that the free energy is convex if we allow for mixing of the pure phases, $-\Phi_0$ and Φ_0.

Problem 1.5 The *mean field approximation* is a concrete procedure to implement the ideas underlying the Landau theory in a specific model. Here we consider the D-dimensional Ising model. Start with the expression for the energy of the Ising model given in section 1.1 and expand the energy in powers of $m_i - \langle m\rangle$, where $\langle m\rangle$ is the expectation value of m_i (to be determined!). In the mean field approximation one neglects al terms of order $(m_i - \langle m\rangle)^2$ and higher. Compare the so obtained effective Hamiltonian obtained in problem **??**.

The partition function for the truncated expression for the energy can then be calculated exactly. Minimizing the corresponding free energy leads to an implicit

equation for $\langle m \rangle$. Determine the critical temperature T_c defined as the maximal temperature for which this equation has a non-trivial solution for $\langle m \rangle$. Expand the free energy in terms of $\langle m \rangle$ for $(T - T_c)/T_c << 1$ up to fourth order in $\langle m \rangle$. Compare the resulting expression with the Landau free energy.

Problem 1.6 The purpose of this problem is to use dimensional arguments in order get a quick derivation of scaling laws of the type found at the end of section 1.4 relating the various critical exponents. The key assumption is that for $t = (T_c - T)/T_c$, the correlation length, $\xi(t) = \xi_0 t^{-\nu}$, is the only scale in the theory. Dimensional counting then implies that

$$\beta \frac{F(T,V)}{V} \propto \xi^{-D}(t)\,,$$

where D is the space dimension. Show that this implies that $c_V \propto |t|^{D\nu-2}$. Combining this with $c_V \propto |t|^{-\alpha}$ (see problem 1.1) we end up with the *Josephson scaling law*

$$2 - \alpha = D\nu\,.$$

Similarly using that for the magnetization

$$B\langle m \rangle = \frac{F_L(T, \langle m \rangle, B) - F_L(T, 0, 0)}{V}\,,$$

show that $\beta(1 + \delta) = D\nu$. There are in total four scaling laws and six critical exponents so that only two of them are independent.

Historical Notes

The idea of the renormalization group was first introduced by by Stueckelberg and Petermann in 1953 and by Gell-Mann and Low in 1954 to cure ultraviolet divergencies in in the perturbative approach to relativistic quantum field theory. We have encountered an example of such a divergence for $\Sigma(0)$ in section 1.7. Renormalization is the procedure to remove such infinities order by order in perturbation theory by adding equally infinite counter terms to the action, that is, the infinities are absorbed through a redefinition of the coupling constants and the fields. The underlying physical intuition is that the original, or "bare" theory is merely a mathematical construct while all physical observables which are expressed solely in terms of renormalised couplings are finite. Generically this renormalization procedure introduces a new scale in the theory. The renormaliztion group then expresses the invariance of the physics of this new scale.

In critical phenomena the renormalization group was first formulated by Wilson in 1971. The purpose here is not to cure divergencies but to reduce the number of degrees of freedom in the effective description of critical phenomena by systematically integrating over short wavelength fluctuations. This is the philosophy that

was followed in this chapter. The physics at large scales should not be affected by this procedure. Thus the sole effect should be a possible redefinition of the dimensionful coupling constants to take the change of scale into account. The renormalization group transformation then relates the parameters of the theory at different steps of this process. The key point in this approach is the existence of fixed points in the renormalization group transformation. At these points the effective theory becomes scale invariant and therefore can describe a physical system at a phase transition.

Since the 1970's there has been a gradual change in philosophy also in quantum field theory as to the interpretation of the renormalized theory. Today one is more inclined to think of the renormalized theory as a long distance approximation to some yet to be discovered unified theory. At long distances the details of this unknown theory are irrelevant, the low energy theory being defined by but a few coupling parameters that define the renormalized theory. The success of Wilson's interpretation of renormalization for critical phenomena surely has contributed to this shift of emphasis.

Further Reading

A classic text book on the theory of phase transitions is H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena*, Oxford University Press (1971). Boundary and surface effects in phase transitions are discussed in S. K. Ma, *Statistical Mechanics* World Scientific (1985). For further discussions of the scaling laws see e.g. K. Huang, *Statistical Mechanics*, John Wiley (1987) or M. Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1991).

Other useful texts on phase transitions and critical phenomena are listed below: L. Landau and E. Lifshitz, *Statistical Physics*, Pergamon Press (1959); L. E. Reichl, *A Modern Course in Statistical Physics*, Arnold (1980); S. K. Ma, *Modern Theory of Critical Phenomena*, N. Goldenfeld, *Lectures on Phase Transitions and The Renormalisation Group*, Addison Wesley (1992), W. A. Benjamin (1976) and at a more advanced level, J.Zinn-Justin, *Quantum Field Theory and Critical Phenomena 2nd ed.*, Oxford University Press (1993); G.Parisi, *Statistical Field Theory*, Addison-Wesley (1988); J. J. Binney, N. J. Dowrick, A. J. Fisher and M. E. J. Newman, *Theory of Critical Phenomena*, Oxford University Press (1992). The experimental values for the critical exponents in table 1.7 were taken from A. Z. Patashinskii and V.L Pokrovskii, *Fluctuation Theory of Phase Transitions*, Pergamon Press (1979).