Ludwig-Maximilians-Universität München
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Lecture Winter term 2018/2019

Conformal Field Theory

Please note: These notes only summarise the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. Changes to this script are made without any further notice at unpredictable times. If you find any typos or errors, please let us know.

Contents (Preliminary)

Reviews.

1. Lecture on Oct. 16. – **Conformal Diffeomorphisms**

- **References**: I do not follow specific books, possible references include [F, R, G, S].
- For most of these lectures we will consider subsets of $\mathbb{R}^n$ with meteric g (mostly g is the Euclidean metric). **Definition** [S]**:** A **conformal diffeomorphism** $\phi\ :\ U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ is a diffeomorphism s.t.

$$(\phi^* g)_p(X,Y) = \Omega^2 g_p(X,Y) \tag{1}$$

  for some function $\Omega : U \to \mathbb{R}_+$.
- The **conformal Killing equation** is the infintesimal version of definition (1). Using **coordinates** $\{x^\mu\}$ this reads

$$x^\mu \mapsto x^\mu + \epsilon f^\mu(x)\,, \qquad \text{with} \qquad f_{\mu;\nu} + f_{\nu;\mu} = \frac{2}{n} f^\alpha{}_{;\alpha} g_{\mu\nu} + O(\epsilon^2)$$

- **Example:** Isometries of $\mathbb{R}^n$ are conformal diffeos with $\Omega \equiv 1$.
- For $n \neq 2$

$$f^\mu(x) = \lambda x^\mu \qquad \text{and} \qquad f^\mu(x) = 2(c\cdot x)x^\mu - (x\cdot x)c^\mu$$

  where $\lambda \in \mathbb{R}_+$ and c^μ is a constant vector. Thus the **conformal algebra** is of dimension $\frac{n}{2}(n+1)(n+2)$.

2. Lecture on Oct., 18. – **Conformal Diffeo's cont'd** [R]

- For the Euclidean metric ($g_{\mu\nu} = \delta_{\mu\nu}$) the condition (1), expressed in coordinates,
$$(\phi^* g)_{\mu\nu} = \delta_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^\mu} \frac{\partial \phi^\beta}{\partial x^\nu} = \Omega^2 \delta_{\mu\nu}$$
makes it explicit that a confomal diffeomorphism is locally equivalent to a rotation followed by a rescaling.
- On can show (problem sheet 1) that the algebra of infinitesimal conformal diffeos is isomorphic to the algebra of isometries, $so(n+1,1)$ of $n+2$ dimensional Minkowski space $\mathbb{R}^{n+1,1}$.
- Similarly, for $g_{\mu\nu} = \eta_{\mu\nu}$ with signature $(1, n-1)$ the algebra of infinitesimal conformal diffeos is isomorphic to the algebra of isometries, $so(n,2)$ of $n+1$ dimensional Anti deSitter space AdS_{n+1}. defined by the hypersurface,
$$X^A X^B \eta_{AB} = -1$$
where, $\eta_{AB} = (+,+,+,+,+.\cdots,+,-,-)$.
- For $n = 2$, identifying $\mathbb{R}^2 \simeq \mathbb{C}$ with coordinate $z = x^1 + ix^2$, condition (1) implies the Cauchy-Riemann equations for $u = \mathrm{Re}(\phi)$ and $v = \mathrm{Im}(\phi)$. Thus, $z \mapsto f(z)$ with $f(z)$ holomorphic in $U \subset \mathbb{C}$ is a conformal diffeo. Similarly, on $\mathbb{R}^2$ with Minkowski metric, any functions $x^+ \mapsto f(x^+)$ and $x^- \mapsto g(x^-)$ define conformal diffeos.
- In order to discuss the global properties of these transformations we consider the **conformal compactification**, adding the point at infity, $\mathbb{C} \cup \{\infty\}$. Globally defined conformal mappings are then of the form
$$f(z) = \frac{P(z)}{Q(z)}$$
where P and Q have at most simple zeros. This defines the group $SL(2,\mathbb{C})$ generated by complex linear combinations of
$$l_1 = -\partial_z. \qquad l_0 = -z\partial_z \qquad \text{and} \qquad l_{-1} = -z^2\partial_z$$
with algebra $[l_m, l_n] = (m-n)l_{m+n}$.

3. Lecture on Oct. 23. – **Conformal Transformations and Critical Phenomena**

- The group $SL(2,\mathbb{C})$ is (perhaps) also familiar from the discussion of the Lorentz group in $3+1$ dimensions where is the group that is generated by the complex generators,
$$T_a = K_a \pm iJ_a$$
where K_a and J_a are the generators of Lorentz boosts and rotations respectively. This the shows that the group $SL(2,\mathbb{C})$ of globally defined conformal transformations in 2 Euclidean dimensions is indeed isomorphic to $SO(3,1)$ as we expect from our discussion in lecture 2 for generic dimensions.
- **Conformal transformations** are defined as the composition of a conformal diffeomorphism (1) and a **Weyl transformation** of the metric g given by
$$\begin{aligned} \mathbb{R}^n \ni p &\mapsto p \\ g &\mapsto \tilde{\Omega}^2 g \end{aligned}$$

and $\tilde{\Omega}$ is chosen such that g is invariant under the composition. More precisely $\Psi = \text{Weyl} \circ \text{conf diff}$, such that

$$\begin{aligned} \Psi : p &\mapsto \phi(p) \\ g &\mapsto \frac{1}{\Omega^2}(\phi^* g) = g \end{aligned}$$

- **Crtitical phenomena** [Go, Sa] such as second order phase transformations are characterised by a few **critical exponents**. For instance, for a ferro magnetic substance we have near the **critical (Curie) temperature**

$$\begin{aligned} magnetisation : \qquad & < M_z > \propto |T - T_c|^{\beta} \\ susceptibility : \qquad & \chi = \frac{\partial < M_z >}{\partial B_z}|_{B=0} \propto |T - T_c|^{-\gamma} \end{aligned}$$

  where B is the external magnetic field and $\beta \simeq 0.33$ and $\gamma \simeq 1.25$ are the **universal** critical exponents.
- In statistical mechanics we one aims to model such critical behaviour with simple **statistical models** captering the degrees of freedom relevant for the universal behaviour. The simplest such model is the **Ising model** where the magnetic spins $\sigma_x \in \{+1, -1\}$ where $x \in \Lambda_{a_0} \subset (a_0\mathbb{Z})^n$ denotes a point in the lattice.
- The relevant observable to compare with experiment is the **correlation function**

(2)
$$< \sigma_x \sigma_y >= \frac{1}{Z} \sum_{\{\sigma_w\}} \sigma_x \sigma_y e^{-\beta H[\{\sigma_w\}, B]}$$

  where

(3)
$$Z = \sum_{\{\sigma_w\}} e^{-\beta H[\{\sigma_w\}, B]}$$

  is the **partition sum**, $\beta = \frac{1}{k_B T}$ is the inverse temperature (NOT TO BE CONFUSED WITH THE CRITICAL EXPONENT) and

(4)
$$\beta H[\{\sigma_w\}, B] = -\beta J \sum_{|x-y|=1} \sigma_x \sigma_y + \beta B \sum_x \sigma_x$$

  is the Ising **Hamiltonian** in the presence of an external magnetic field B.

4. Lecture on Oct. 25. – **RG Flow**

- In Kadanoff's **block spin** method one replaces each individual spin σ_x by the average

$$\sigma_X = \frac{1}{m_L} \sum_{x \in X} \sigma_x \,, \qquad m_L = \sum_{x \in X} < \sigma_x >$$

  over the block X of length La_0 containing L^n spins. The model is said to be **renormaliseable** if the description in terms of the lock spins is given in terms of the **same** Hamiltonian, possibly with renormalised couplings

$$g^{(L)} = U_L(g, h) \,, \qquad h^{(L)} = V_L(g, h)$$

  where $g = \beta J$ and $h = \beta B$.

- This blocking is the repeated until one encounters a **critical point** (g^*, h^*) (assuming it exists), where

$$g^* = U_L(g^*, h^*)\,, \qquad h^* = V_L(g^*, h^*)$$

This point should correspond to the critical temperature for the original ferro magnet, characterised by the absence of any scale.

- Due to the absence of scale at the critical point it is legitimate to replace the lattice model by a suitable continuum field theory (zero lattice constant), still to be defined. Similarly, we can replace L by a continuous variable. The logarithmic derivative

$$L\frac{d}{dL}g^{(L)} =: u(g^{(L)})$$

is then a function that depends only on $g^{(L)}$ (assuming $h^{(L)} = 0$ for simplicity) and that vanishes at g^*. At linear order

$$u(g^{(L)}) = (g^{(L)} - g^*)y\,, \qquad y = \frac{\partial u}{\partial g}|_{g=g^*}$$

we find for the **correlation length** ξ with $\xi(g^{(L)}) = \frac{1}{L}\xi(g^{(L)})$ on dimensional grounds, the critical behaviour

(5)
$$\xi(g^{(L)}) \propto (g^{(L)} - g^*)^{-\frac{1}{y}} \propto t^{-\frac{1}{y}}$$

where $t = \frac{T-T_c}{T_c}$ is the **reduced temperature**.

5. Lecture on Oct. 30. – **Continuum Description**

- In order to compare the lattice model with the continuum field theory we consider the **scaling limit**

$$< \sigma_x \sigma_y >^* := \lim_{\lambda\to\infty} \lambda^{(n-2+\eta)} < \sigma_{\lambda x}\sigma_{\lambda y} >$$

which is insensitive to the details inside the blocks X.

- The simplest continuum theory is the **Gaußian model** with euclidean action

$$S[\varphi] = \frac{1}{2}\int \left((\partial\varphi)^2 + m^2\varphi^2\right) d^n x$$

and correlation function

$$C(x,y) := < \varphi(x)\varphi(y) > \propto \frac{e^{-|m||x-y|}}{|x-y|^{n-2}}$$

that agrees with $< \sigma_x\sigma_y >^*$ for $m = 0$ and $\eta = 0$. For $m \neq 0$ this model predicts a susceptibility

$$\chi \propto \int d^n x\; C(x) \propto m^2.$$

- The only scale present is m^2 which, following Landau's description of phase transitions, we identify with t. This then predicts the critical exponent $\gamma = 1$ which is still far from the experimental value. The reason for this is that in Landau's description the fluctuations are ignored.

6. Lecture on Nov. 6. – **Renormalisation**

- Renormalisability and symmetry don't allow for many more models except

$$S[\varphi] = \frac{1}{2}\int \left((\partial\varphi)^2 + m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 \right) d^n x$$

Since this model is non-linear the fluctuations renormalise both, λ and m. This can be seen by calculation the loop-corrected 4-point vertex up to one loop [ID],

$$\Gamma_4 = \lambda - \frac{3}{2}\lambda^2 \int d^n q \frac{1}{(q^2+m^2)^2} = \lambda - \frac{3}{2}\lambda^2 (m^2)^{-\frac{\epsilon}{2}} \frac{A}{\epsilon}\,, \qquad \epsilon = 4-n$$

where $A = 2\frac{\Gamma(3-\frac{n}{2})}{(2\sqrt{\pi})^n}$. To continue we write

$$\lambda = (\lambda^{(0)} = g_R (m^2)^{\frac{\epsilon}{2}}) + \lambda^{(1)}$$

where u_R is the dimensionless, **renormalized coupling** and λ is the **bare coupling**. We then demand that Γ_4 agrees with $\lambda^{(0)}$ and thus

$$\lambda = \left(g_R + \frac{3}{2} g_R^2 \frac{A}{\epsilon} \right) (m^2)^{\frac{\epsilon}{2}}$$

or, equivalently,

$$g_R = \frac{\lambda}{(m^2)^{\frac{\epsilon}{2}}} - \frac{3}{2}\frac{\lambda^2}{(m^2)^{\epsilon}}\frac{A}{\epsilon}$$

Since the bar coupling λ is independent of m we the the find

$$\beta(g_R) = m^2 \frac{d}{dm^2} g_R = -\frac{\epsilon}{2}\left(g_R - \frac{3}{2} g_R^2 \frac{A}{\epsilon} \right)$$

The critical point is then given by either $g_R = 0$ (Gaußian model) or $g_R = \frac{2\epsilon}{3A}$ (**Wilson Fisher fixed point**).

Similarly the mass is renormalised as seen be computing

$$m_R^2 = \Gamma_2 = m^2 + \lambda (m^2)^{\frac{n-2}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{(2\sqrt{\pi})^n}$$

With the ansatz $m_R^2 = (m^2)^{1+c(\lambda)}$ we find from the above

(6)
$$c(\lambda) = \frac{A}{4}\lambda^2 (m^2)^{-\frac{\epsilon}{2}} = g_R^* \frac{A}{2} = \frac{\epsilon}{6}$$

For $n = 3$ ($\epsilon = 1$) this gives the critical exponent $\gamma \simeq 1.2$ which is already close to the measured value 1.25 .

- More generally, comparing (11) with (5) we find $y \simeq 1 - \frac{\epsilon}{12}$ in this model.

7. Lecture on Nov. 8. – **Anomalous Dimensions and Primary Fields**

- To summarize, taking into account fluctuations at first order in λ we found an anomalous scaling of the correlation length *near* the Wilson-Fisher critical point. At the critical point the 2-point correlation function scales as

$$C(x,y) = < \varphi(x)\varphi(y) > \propto \frac{1}{|x-y|^{n-2}}$$

while for the for the lattice model one furthermore finds an **anomalous dimension** for φ,

$$C(x,y) \propto \frac{1}{|x-y|^{2\Delta}}$$

with $2\Delta = n - 2 + \eta$. A non-vanishing value for η in the continuum theory arises at second order in λ [ID].

- The **Källen Lehman representation** of the 2pt correlator

$$\Delta(p) = \int e^{ipx} C(x) d^n x = \int e^{ipx} \frac{1}{|x|^{2\Delta}} d^n x = \frac{(4\pi)^{n/2}}{4^{2\Delta}\Gamma(2\Delta)\Gamma(2\Delta - \frac{n}{2} + 1)} \int_0^\infty \frac{\mu^{2\Delta - \frac{n}{2}}}{\mu + p^2} d\mu$$

shows that at the interacting fixed point ($\eta \neq 0$) the perturbative spectrum of the theory contains a continuum of states with $m^2 = \mu$ rather than a discrete set of states.

- If

$$< \varphi_\Delta(x)\varphi_\Delta(y) > \propto \frac{1}{|x-y|^{2\Delta}} \tag{7}$$

we say that Δ is the **weight** of the field φ.

- A field φ is said to be a **primary field of weight** Δ if φ transforms like a tensor field under translations and rotations (or Lorentz transformations) and furthermore

$$\varphi(x) = \lambda^\Delta \tilde{\varphi}(\tilde{x})$$

under dilataions, $\tilde{x} = \lambda x$. In particular, for $\Delta = 0$, φ is a **scalar field** (a function).

- The notion of a primary field can be extended to the whole conformal group (including special conformal transformations) by embedding the space $\mathcal{F}(\mathbb{R}^n)$ functions on $\mathbb{R}^n$ in the space $\mathcal{F}(\mathbb{R}^{n+2})$ functions on $\mathbb{R}^{n+2}$ where conformal transformations are represented linearly by an $SO(n+1,1)$ matrix Λ, followed by a dilatation, D, to bring the point back to $\mathbb{R}^n$ (realised as a section on the light cone). Concretely,

$$\varphi(x^\mu) \hookrightarrow \phi(X^A) \stackrel{\Lambda}{=} \tilde{\phi}(\tilde{X}^A) \stackrel{D}{=} \lambda^\Delta \tilde{\tilde{\phi}}(\tilde{\tilde{X}}^A) \stackrel{restr. to\ \mathbb{R}^n}{\to} \lambda^\Delta \tilde{\varphi}(f^\mu(x))$$

where $x^\mu \mapsto f^\mu(x)$ is a conformal mapping.

- If the statistical measure (or vacuum state for Minkowski CFT, see next lecture) is invariant under conformal transformations then we conclude that under the conformal transformation $x \mapsto \tilde{x} = f(x)$ the 2 pt correlator transforms like

$$< \tilde{\varphi}(\tilde{x}^\mu)\tilde{\varphi}(\tilde{y}^\mu) > = \lambda^{-2\Delta} < \varphi(x^\mu)\varphi(y^\mu) >$$

where $\lambda = \frac{1}{n} f^\mu_{,\mu}$ in agreement with (7).

- By the above embedding and assuming conformal invariance one finds similarly

$$< \phi_{\Delta_1}(X^A)\phi_{\Delta_3}(Y^B)\varphi_{\Delta_3}(Z^C) > = \frac{c}{(X\cdot Y)^a (X\cdot Z)^b (Y\cdot Z)^c}$$

with $a + b = \Delta_1$, etc. from scale invariance.

8. Lecture on Nov. 13. – **Scale vs. Confomal Invariance**

- A local Poincaré invariant field theory has a symmetric, conserved stress tensor $T_{\mu\nu}$. If the model is scale-invariant then

$$T^\mu_\mu = \partial_\mu V^\mu$$

holds for some local field (operator) V^μ.
- In Lorentzian signature $T_{\mu\nu}$ gives rise to a family of Noether charges

$$J_\mu = \int_\Sigma \left(\xi^\nu T_{\mu\nu} - \frac{1}{n} \xi^\nu_{,\nu} V_\mu \right) \, d^{n-1}x$$

where ξ^μ is the Killing field.
- In the **canonical formulation** the transformation of primary field correlators is then given by

$$\delta_\xi < \varphi(x^\mu) \cdots \varphi(y^\mu) > = < [J^0, \varphi(x^\mu) \cdots \varphi(y^\mu)] >$$

which vanishes if the vacuum is invariant.

9. Lecture on Nov. 15. – **Special Conformal Transformations**

- Note that the term in J^μ containing V^μ implements the Weyl transformation on the field φ while the term containing $T_{\mu\nu}$ implements the diffeomorphism part. For instance, for dilatations,

$$\delta \varphi_\Delta(x) = -i(x^\mu \partial_\mu - \Delta) \varphi_\Delta(x) \tag{8}$$

- If ξ^μ parametrises a special conformal transformation the corresponding Noether charge $J^0(\xi)$ is conserved if and only if there is a local field $L^{\mu\nu}$ such that

$$V^\mu = \partial_\nu L^{\mu\nu}$$

- For $n = 2$ scale invariance and unitarity implies the existence of $L^{\mu\nu}$ while for $n > 2$ the precise necessary conditions for this to hold are still not completely settled.

10. Lecture on Nov. 20. – **Radial Quantisation**

- A quantum state in QFT amounts to specifying the wave function on a Cauchy hypersurface, Σ_t. After a **Wick rotation** of time, $t - i\tau$, we obtain a field theory defined on $\mathbb{R}^n$ with Euclidean signature and hypersurfaces, Σ_τ.
- Using invariance of a CFT under **inversion** $\Sigma_{-\infty}$ is mapped to the origin of $\mathbb{R}^n$ while on Σ_0 the inversion maps the origin in Σ_0 to the point at infinity. Adding this point we obtain the **conformal compactification** of Σ_0 which through stereographic projection is isomorphic to the unit sphere S^{n-1}.
- In a CFT it is convenient to replace the hypersurfaces Σ_τ by concentric spheres S^{n-1}_r of radius r related to $S^{n-1}_{r'}$ by a dilation. Thus the dilatation operator D plays the role of the Hamiltonian in a CFT. Furthermore, instead of defining a state ψ on Σ_t one defines an (abstract) state ϕ on S^{n-1}_r. This is usually referred to **radial quantisation**.
- Since S^{n-1}_r can be shrunk to a point by a suitable dilation, one expects that a state ϕ should correspond to the insertion of a suitable primary field (or operator) at the origin of $\mathbb{R}^n$. This is the **state operator correspondence**.

- **Operator state correspondence:** Conversely, any primary field $\phi_\Delta(x)$ inserted at $x=0$, defines an eigenstate of D. Indeed, from (8) we have

$$D|\Delta> \equiv [D, \phi_\Delta](0) = i\Delta|\Delta>$$

- Furthermore, from $P_\mu \sim -i\partial_\mu$ we conclude that $P_\mu|\Delta>$ has dimension $\Delta+1$,

$$P_\mu|\Delta> = [P_\mu, \phi_\Delta](0) = |\Delta+1>$$

More generally,

$$|\Delta> \stackrel{P_\mu}{\to} |\Delta+1> \stackrel{P_\mu}{\to} |\Delta+2> \cdots$$

generates a module of **descendants** for each primary state $|\Delta>$. Similarly, with

$$K_\mu \sim -i\left(-2x_\mu\Delta + 2x_\mu(x^\alpha\partial_\alpha) - x^2\partial_\mu\right)$$

one finds the the sequence

$$0 \stackrel{K_\mu}{\leftarrow} |\Delta> \stackrel{K_\mu}{\leftarrow} |\Delta+1> \stackrel{K_\mu}{\leftarrow} |\Delta+2> \cdots$$

11. Lecture on Nov. 22. – **Inner Product & Unitarity Bounds**

- In a unitary CFT it is possible to define a non-degenerate positive definite in product by glueing the ball bounded by the unit sphere S^{n-1} with an operator $\mathcal{O}$ (primary field or descendent) inserted at the origin with another ball with operator $\mathcal{O}'$ inserted along S^{n-1}. The result is the two point function

$$\begin{aligned} < \Delta'|\Delta> &:= \lim_{x\to 0} < (I^*\mathcal{O}_{\Delta'})(x)\mathcal{O}_\Delta(x)> \\ &= \delta_{\Delta',\Delta} \lim_{x\to 0}|x|^{-2\Delta} < \mathcal{O}_{\Delta'}(I(x))\mathcal{O}_\Delta(x)> = const. \delta_{\Delta',\Delta} \end{aligned} \tag{9}$$

The *const*. can be set to one bey a suitable rescaling of $\mathcal{O}$.

- The adjoint of P_μ w.r.t. to this inner product is easily found noting that

$$<\Delta| = I(|\Delta>)$$

Then,

$$P_\mu^\dagger = K_\mu$$

Similarly, $D^\dagger = -D$.

- The **unitarity bounds** are consequences of these relations. In particular, if we denote by $|\Delta,\ell>$ a **highest weight** in the spin ℓ representation of $SO(n)$, then the condition

$$0 < \quad ||P_\mu P_\nu|\Delta>||^2 = <\Delta|K_\mu\, K_\nu P_\mu P_\nu|\Delta>$$

implies,

$$\Delta(\ell) \geq \ell + n - 2 \quad \text{for} \quad \ell > 0 \quad \text{and} \quad \Delta \geq \frac{n}{2} - 1 \quad \text{for} \quad \ell = 0\,. \tag{10}$$

Furthermore holds iff $\mathcal{O}_{\Delta,\ell}(x)$ satisfies a conservation equation, eg. $\Box\mathcal{O}_\Delta(x) = 0$ for $\ell = 0$, $\partial^\mu(\mathcal{O}_{\Delta,1})_\mu(x) = 0$ for $\ell = 1$ etc.

12. Lecture on Nov. 27. – **Operator Product Expansion**

- Inserting a primary field ϕ_1 at the origin and another primary field ϕ_2 at some point $x \neq 0$ defines some state and thus, by the operator state correspondence must be equivalent a linear combination of primaries and descendents inserted at the origin. Furthermore, since in a Hilbert space Cauchy series converge, this expansion is convergent. Thus we have the **operator product expansion (OPE)**
$$\phi_{\Delta_1}(x)\phi_{\Delta_2}(0) = \sum_{\mathcal{O},primary} C^{12}_{\mathcal{O}}(x,\partial_y)\mathcal{O}(y)|_{y=0}$$
where $C^{12}_{\mathcal{O}}(x,\partial_y)$ with polynomial dependence on ∂_y is a generating function for the descendants.
- For $\Delta_1 = \Delta_2 = \Delta$ one shows ([R, DO] and exercise sheet), upon acting with the generators D and K_μ on this equation, that
$$C^{12}_{\mathcal{O}}(x,\partial_y) = \lambda_{12\mathcal{O}}\frac{1}{|x|^{2\Delta-\Delta_{\mathcal{O}}}}C_{\mathcal{O}}(x,\partial_y) \tag{11}$$
where, up to an overall normalisation, $\lambda_{12\mathcal{O}}$ and $C_{\mathcal{O}}(x,\partial_y)$ are completely determined by the 3-point functions
$$< \phi_\Delta(x_1)\phi_\Delta(x_2)\mathcal{O}(x_3) >= \frac{\lambda_{12\mathcal{O}}}{|x_1-x_2|^{2\Delta-\Delta_{\mathcal{O}}}|x_1-x_3|^{\Delta_{\mathcal{O}}}|x_2-x_3|^{\Delta_{\mathcal{O}}}}$$
$C_{\mathcal{O}}(x,\partial_y)$ is of the form,
$$C_{\mathcal{O}}(x,\partial_y)(1 + c_k(x)^k(\partial)^k)$$
where the c_k are depend on $\Delta_{\mathcal{O}}$ only (see exercise).
- **Partial wave expansion:** We can substitute the OPE in the 4-pt function twice to find
$$\begin{aligned} &< \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4) >= \\ &\sum_{\mathcal{O},\mathcal{O}',primary} C^{\Delta\Delta}_{\mathcal{O}}(x_1-x_2,\partial_y)C^{\Delta\Delta}_{\mathcal{O}'}(x_3-x_4,\partial_{y'}) < \mathcal{O}(y)\mathcal{O}(y') > |_{y=x_2,y'=x_4} \end{aligned} \tag{12}$$
This is the **conformal partial wave expansion** of conformal correlation functions.

13. Lecture on Nov. 29. – **Conformal Bootstrap**

- So far this procedure leads to a definite result for N-pt functions for any set of $\{\Delta_i, \lambda_{ijk}\}$. However, there constraints implied by the requirement that replacing the $(1-2),(3-4)$ OPE by $(1-4),(3-2)$ leads to the same 4-pt function (crossing symmetry).
- **Crossing symmetry:** While 2- and 3-pt functions are completely determined by conformal symmetry this is not so for the 4-pt function. The general conformally invariant expression for the 4-pt function takes the from
$$< \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4) >= \frac{f(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}$$
where
$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \qquad u = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

are the invariant **conformal cross ratios**. Crossing symmetry requires invariance under $x_1 \leftrightarrow x_2$ or, equivalently,

$$\left(\frac{v}{u}\right)^{\Delta} f(u,v) = f(v,u)$$

- On the other hand, using (11) and (12) we can write the 4-pt functions as

$$< \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4) >= \sum_{\mathcal{O},primary} \lambda^2_{\Delta\Delta\mathcal{O}} G_{\mathcal{O}}(u,v)$$

where the **conformal block** $G_{\mathcal{O}}(u,v)$ accounts for the contribution of $\mathcal{O}$ and its descendants to the 4-pt function. The crossing symmetry condition can then equivalently be written as

$$\sum_{\mathcal{O},primary} \lambda^2_{\Delta\Delta\mathcal{O}} \left(v^\Delta G_{\mathcal{O}}(u,v) - u^\Delta G_{\mathcal{O}}(v,u)\right) = 0$$

- **Def: CFT:** A CFT is a set of data $\{\Delta_i, \lambda_{ijk}\}$ such that crossing symmetry holds for the 4-pt function.
- The goal of the **conformal bootstrap** is to find $\{\Delta_i, \lambda_{ijk}\}$ such that this equation is satifield.
- **Rem:** Crossing symmetry is equivalent to demanding the the OPE is **associative**. Once the constraints are satisfied for the 4-pt function it is not hard to see that no further constraints are imposed by higher -pt functions.
- **AdS/CFT correspondence:** The isometry group of (Euclidean) AdS_{n+1} is $SO(n+1,1)$. The **Poincaré patch** of AdS_{n+1} is parametrised by the coordinates $(z, \underline{x} \in \mathbb{R}^n)$ with metric

$$ds^2 = \frac{dz^2 + d\underline{x}^2}{z^2}$$

On the conformal boundary of $(z \to 0)$ of the Poincaré patch the $SO(n+1,1)$ generators reduce the conformal Killing fields on $\mathbb{R}^n$ (see exercise).
- Any correlation function of an (not necessarily conformally invariant) AdS-invariant field theory, evaluated on the conformal boundary, is the consistent with conformal symmetry on $\mathbb{R}^n$. Then, furthermore summing over s, t and u-channels to enforce crossing symmetry this produces solutions to the conformal bootstrap equations. This the AdS/CFT correspondence.

14. Lecture on Dec. 4. – **CFT in 2 Dimensions**

- Rep: in 2 dimensions any holomorphic function $f(z)$ solves the conformal Killing equation. A basis for infinitesimal conformal transformations (Killing field) is the given by $\{l_n = \xi^z_n \partial_z = z^{n+1}\partial_z\}$ with $[l_n, l_m] = (n-m)l_{n+m}$. This is the **deWit algebra**. The representation L_n of l_n in field theory is given in terms of the **moments**

$$L_n = T(\xi_n) = \frac{1}{2\pi i}\oint dz\, \xi^z_n\, T_{zz} = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T_{zz}$$

and similarly for $\bar{L}_n$. The hermitian conjugate (exercise) of L_n is given by

$$L_n^\dagger = L_n$$

From

$$\delta_\xi \phi_\Delta(w) = [T(\xi), \phi_\Delta(w)] \overset{rad.\ quant.}{=} \frac{1}{2\pi i}\oint dz\, \xi^z\, T_{zz}\phi_\Delta(w)$$

for primary fields $\phi_\Delta = \phi_{h\bar{h}}$ we then infer the OPE $T(z) \equiv T_{zz}$

(13) $$T(z)\phi_\Delta(w) = \frac{h}{(z-w)^2}\phi_\Delta(w) + \frac{1}{(z-w)}\partial_w\phi_\Delta(w) \qquad + \quad \text{regular terms}$$

The OPE of $T(z)$ with itself on the other hand, should be given by (on dimensional grounds)

$$T(z)T(w) = \frac{c\mathbb{I}}{2(z-w)^4}T(w) + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w) + \quad \text{reg.}$$

with $c > 0$ for a unitary theory (positive definite inner product). This, then leads to

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c\mathbb{I}}{12}(n^3 - n)\delta_{n+m,0}$$

The term proportional to $\mathbb{I}$ is a **central extension** since it commutes all other generators.

- The centrally extended deWit algebra is the **Virasoro algebra**.
- As a consequence of the central term in the OPE of $T(z)$ with itself T does not transform as tensor conformal transformations. Rather we have under $z \to z + \epsilon\xi(z)$,

$$\delta_\epsilon T(z) = \epsilon\xi^z\partial_z T(z) - 2\epsilon(\partial_z\xi^z)T(z) + \frac{c}{12}\epsilon(\partial_z^3\xi^z) + O(\epsilon^2)$$

or, for a finite transformation, $\tilde{z} = f(z)$,

$$T(z) = \frac{1}{(\partial_z f(z))^2}\tilde{T}(f(z)) + \frac{c}{12}\{f, z\}$$

where

$$\{f, z\} = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2$$

is the Schwarzian derivative.

15. Lecture on Dec. 6. – **Highest Weight Representations**

- For $\phi_\Delta = \mathbb{I}$, which corresponds to the vacuum $|0>$ through the operator state correspondence we have $T(z)\phi_\Delta = T(z)$. Integration against $dz\ z^{n+1}$ the gives

$$L_n|0>= 0\,, \quad n \geq -1$$

and similarly for $\bar{L}_n$. Then, since $\{L_1, L_0, L_{-1}\}$ forms a closed subalgebra (isomorphic to $sl(2,\mathbb{R})$), the vacuum state is invariant under $sL(2,\mathbb{C}) = sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$.

- More generally, for a primary field ϕ_Δ and its corresponding state $|\Delta>$ it follows from (13) that

$$L_n|\Delta>= 0\,, \quad n \geq 1 \quad \text{and} \quad L_0|\Delta>= h|\Delta>$$

Thus, primary field generate **highest weight** states $|\Delta>$ with weight h (and $\bar{h}$) and $L_n|\Delta>$ is a **desendent** of $|\Delta>$.

- An import result that follows form positivity of the inner product is that in a unitary CFT any state $|\psi>$ is a liner combination of highest weight states and their descendants. As a consequence the Hilbert space $\mathbb{H}$ of a CFT has an orthogonal decomposition of the form

$$\mathbb{H} = \oplus_{h,\bar{h}}\mathcal{V}_h \otimes \mathcal{V}_{\bar{h}}$$

where $\mathcal{V}_h$ consists of a highest weight state $|\Delta >$ and its descendants. $\mathcal{V}_h$ is a **Verma module**.

- In a unitary representation of the Virasoro algebra (CFT) it holds that $h, \bar{h} \geq 0$.
- In a unitary representation of the Virasoro algebra (CFT) it holds that $c > 0$.
- **Virasoro characters:** The **character** of a Verma module is defined as

$$\chi(c,h) = \mathrm{Tr}_{\mathcal{V}_{c,h}}(q^{L_0 - \frac{c}{12}})\,, \qquad q = e^{2\pi i \tau}$$

with $Im(\tau) > 0$.

16. Lecture on Dec. 13. – **Unitarity** [F, QJ]

- In contrast to higher dimensional CFT's the dimension of the subspaces in a Verma module creases with the **level** (that is the subspaces of descendants of fixed conformal weight) and we cannot exclude that certain linear combinations of descendants of fixed conformal weight have negative norm. In order to analyse this issue we consider the hermitian matrices, $M^{(k)}_{ij} =< i|j >$, of inner products at fixed level k. Unitarity then requires that $M^{(k)}_{ij}$ is positive definite and, in particular, $\det(M^{(k)}) > 0$.
- If $\det(M^{(k)}) = 0$ for some k, then $\mathcal{V}_h$ contains a **null state** (i.e. orthogonal to all states including itself) which can be identified with zero and which removes it form $\mathcal{V}_h$. If $\det(M^{(k)}) < 0$ then the module contains a negative norm state which cannot be removed.
- For $c \geq 1$ there are no states of negative norm.
- Forthermore, it can be shown that except for a discrete set of central charges and conformal weight $\{h_{rs}(c)\}$ all Verma modules contain negative norm states at some level. The models with no negative norm states are given by

$$\begin{aligned}
c &= 1 - \frac{6}{m(m+1)}\,, \quad m = 2, 3, \cdots \\
h_{rs} &= \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \\
&= \frac{1}{24}(c-1) + \frac{1}{4}\left(r\frac{\sqrt{1-c} + \sqrt{25-c}}{\sqrt{24}} + s\frac{\sqrt{1-c} - \sqrt{25-c}}{\sqrt{24}} \right) \\
&=: h_0 + \frac{1}{4}\left(r\alpha_+ + s\alpha_-\right)\,, \quad 1 \leq r < m\,, \quad 1 \leq s \leq r
\end{aligned} \tag{14}$$

17. Lecture on Dec. 18. – **BPZ Equations** [F]

- The Verma modules $\mathcal{V}_{h_{12}}$ and $\mathcal{V}_{h_{21}}$ contain a **null state** $|\chi >= (L_{-2} + \eta L^2_{-2})|h >$ for $\eta = -\frac{3}{2(2h+1)}$. This implies, in turn that

$$0 = \left\{ \sum_{i=1}^{P} \frac{1}{z - w_i}\partial_{w_i} + \frac{h_i}{(z-w_i)}^2 + \eta\partial_z^2 \right\} < \phi_h(z)\psi_1(w_1)\cdots\psi_P(w_P) >$$

where $\phi_h(z)$ is the primary field corresponding to $|h >$.
- Applying this to the 3-point function

$$< \phi_h(z)\phi_{h_1}(w_1)\phi_{h_2}(w_2) >= \frac{\lambda_{h12}}{(z-w_1)^{h+h_1-h_2}(z-w_2)^{h+h_2-h_1}(w_1-w_2)^{h_1+h_2-h}}$$

one finds that λ_{h12} can be non-zero only if

$$h_2 = h_0 + \frac{1}{4}\alpha_2^2$$

where

$$\alpha_2 = \alpha_1 \pm \begin{cases} \alpha_+ & h = h_{21} \\ \alpha_- & h = h_{12} \end{cases} \tag{15}$$

and α_1 is defined through $h_1 = h_0 + \frac{1}{4}\alpha_1^2$. These **selection rules** thus are a consequence of the presence of a null state.

- In terms of the OPE this means that

$$\begin{aligned} \phi_{h_{21}} \times \psi_{(\alpha)} &\sim \psi_{(\alpha+\alpha_+)} + \psi_{(\alpha-\alpha_+)} \\ \phi_{h_{12}} \times \psi_{(\alpha)} &\sim \psi_{(\alpha+\alpha_-)} + \psi_{(\alpha-\alpha_-)} \end{aligned} \tag{16}$$

18. Lecture on Dec. 20. – **Ladder Operators** [F]

- For $\psi_{(\alpha)} = \phi_{rs}$ we then have the **Fusion Rules**

$$\begin{aligned} \phi_{21} \times \phi_{rs} &\sim \phi_{r+1,s} + \phi_{r-1,s} \\ \phi_{12} \times \phi_{rs} &\sim \phi_{r,s+1} + \phi_{r,s-1} \end{aligned} \tag{17}$$

which shows that ϕ_{21} and ϕ_{12} act as ladder operators for r and s respectively. Generically fusion generates an infinite family of Verma modules.

- However, if there exist $p, p' \in \mathbb{Z}$ such that $p\alpha_- + p'\alpha_+ = 0$ (=**rational CFT's**) then there is some redundancy in the rs parametrisation. In particular,
  1. $h_{r,s} = h_{r+p',s+p}$
  2. $h_{r,s} = h_{p'-r,p-s}$
  3. $h_{r,s} + rs = h_{p'+r,p-s} = h_{p'-r,p+s}$
  4. etc.

  Consequently, there are infinitely many sub modules (null vectors) in each Verma module $\mathcal{V}_{h_{rs}}$ which, in turn, imply infinitely many BPZ equations and thus infinitely many constraints on the OPE coefficients.
- Rational minimal models are conveniently parametrised by

$$c = 1 = \frac{6(p-p')^2}{pp'}, \qquad h_{rs} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$$

with $p = p' + 1$.

19. Lecture on Jan. 8. – **Ising model**[F]

- For $p' = 1$, $c = -2$ the minimal model is not unitary since $c < 0$.
- For $p' = 2$, $c = 0$ all highest weight states have zero norm.
- For $p' = 3$, $c = \frac{1}{2}$ an independent set of highest weighs states is given by

$$\begin{aligned} \mathbb{I} &:= \phi_{(1,1)} \quad ; \quad h = 0 \\ \sigma &:= \phi_{(1,2)} \quad ; \quad h = \frac{1}{16} \\ \epsilon &:= \phi_{(1,2)} \quad ; \quad h = \frac{1}{2} \end{aligned} \tag{18}$$

with OPE

$$\begin{aligned} \sigma\times\sigma &\sim \mathbb{I}+\epsilon \\ \sigma\times\epsilon &\sim \sigma \\ \epsilon\times\epsilon &\sim \mathbb{I} \end{aligned} \tag{19}$$

- ϵ can be identified with a free Majorana-Weyl fermion ψ in 2 dimensions while σ can be only expressed non-locally in terms of ψ through bosonisation.
- Since $\psi(z)\times(0)\sim\frac{1}{z}$ the modes $\{\psi_n\}$ of ψ form a centrally extended commutative symmetry analogous to the Virasoro algebra represented on σ/ This explains the structure of the OPE algebra (19).
- The fields σ and ϵ can be identified with the order parameter (magnetisation) and energy density of the continuum limit of the Ising model at the critical point.

20. Lecture on Jan. 10. – **Ising model, Potts model**[F]

- The minimal model with $p'=4$, $c=\frac{7}{10}$ primary fields

$$\begin{aligned} \mathbb{I} &:= \phi_{(1,1)} \quad ; \quad h=0 \\ \epsilon &:= \phi_{(1,2)} \quad ; \quad h=\frac{1}{10} \\ \epsilon' &:= \phi_{(1,3)} \quad ; \quad h=\frac{3}{5} \\ \epsilon'' &:= \phi_{(1,4)} \quad ; \quad h=\frac{3}{2} \\ \sigma &:= \phi_{(2,2)} \quad ; \quad h=\frac{3}{80} \\ \sigma' &:= \phi_{(2,4)} \quad ; \quad h=\frac{7}{16} \end{aligned} \tag{20}$$

  describes the tri-critical point of the dilute Ising model.
- The OPE algebra

$$\begin{aligned} \epsilon\times\epsilon &\sim \mathbb{I}+\epsilon' \\ \epsilon\times\epsilon' &\sim \epsilon+\epsilon'' \\ \epsilon\times\epsilon'' &\sim \epsilon' \\ \epsilon'\times\epsilon' &\sim \mathbb{I}+\epsilon' \\ \epsilon'\times\epsilon'' &\sim \epsilon \\ \epsilon''\times\epsilon'' &\sim \mathbb{I} \\ \epsilon\times\sigma &\sim \sigma+\sigma' \\ \epsilon\times\sigma' &\sim \sigma \\ \epsilon'\times\sigma &\sim \sigma+\sigma' \\ \epsilon'\times\sigma' &\sim \sigma \\ \epsilon''\times\sigma &\sim \sigma \\ \epsilon''\times\sigma' &\sim \sigma' \\ \sigma\times\sigma &\sim \mathbb{I}+\epsilon+\epsilon'+\epsilon'' \\ \sigma\times\sigma' &\sim \epsilon+\epsilon' \\ \sigma'\times\sigma' &\sim \mathbb{I}+\epsilon'' \end{aligned} \tag{21}$$

again reveals the presence of an extra symmetry algebra generated by ϵ'' with Virasoro conformal weight $\frac{3}{2}$. $[\mathbb{I},\epsilon'']$ and $[\epsilon,\epsilon']$ transform as doublets under ϵ'' while σ and σ' transform as singlets.

- This is compatible with the superconformal algebra

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c\mathbb{I}}{12}(n^3-n)\delta_{n+m,0} \\ \{G_n, G_m\} &= 2L_{n+m} + \frac{c\mathbb{I}}{3}(n^2-\frac{1}{4})\delta_{n+m,0} \end{aligned}$$

$$[L_n, G_m] = (\frac{n}{2}-m)G_{n+m} \tag{22}$$

and indeed the $c = \frac{7}{10}$ is the only model which is minimal w.r.t. to both, the Virasoro- and the superconformal algebra.

- The minimal model with $p' = 5$, $c = \frac{4}{5}$ has a closed subset of primary fields

$$\begin{aligned} \mathbb{I} &:= \phi_{(1,1)} \quad ; \quad h = 0 \\ \epsilon &:= \phi_{(2,1)} \quad ; \quad h = \frac{2}{3} \\ X &:= \phi_{(3,1)} \quad ; \quad h = \frac{7}{5} \\ Y &:= \phi_{(4,1)} \quad ; \quad h = 3 \\ \sigma &:= \phi_{(3,3)} \quad ; \quad h = \frac{1}{15} \\ \sigma' &:= \phi_{(4,3)} \quad ; \quad h = \frac{2}{3} \end{aligned}$$

whose OPE algebra includes $Y \times Y \sim \mathbb{I}$. The presence of a dimension 3 (or more generally half integer weight) with this OPE signals the presence of an extra symmetry for which this field is the conserved current. This is the W_3 algebra (see [BS]) for a review) which contains the Virasoro algebra just like the superconformal algebra contains the Virasoro algebra for $p' = 4$. The 3-state potts model is the only model which is at the same time a Virasoro minimal model and a W_3 minimal model.

21. Lecture on Jan. 15. – **Landau Ginzburg description**[F]

- It turns out that the diagonal (p, p') minimal models have a Lagrangian description as the critical point (RG-fixed point) of a Landau Ginzburg effective model with Lagrangian

$$S[\Phi] = \int d^2x \left((\partial\Phi)^2 + :\Phi^{2(p'-1)}: \right)$$

where $\Phi = \phi_{(2,2)}$. The normal ordering is defined by subtracting the most singular term in the OPE, eg.

$$\Phi_2 \equiv :\Phi^2: = \lim_{|z|\to 0}\left[(|z|^2)^{2h_\Phi - h_{\Phi_2}}\Phi(z,\bar{z})\Phi(0,0) - \frac{\mathbb{I}}{(|z|^2)^{2h_\Phi}} \right]$$

and similarly, by recursion for the higher order terms. This series of composite operators stops at $:\Phi^{2m-3}: = \partial^2\Phi$ by the equation of motion. In this way the Landau Ginzburg at the RG-fixed point reproduces all primary fields as composites $:\Phi^k:$, $k = 1, \cdots 2p' - 4$.

22. Lecture on Jan. 17. – **Boundary Conformal Field Theory** [F]

- We will consider a conformal field theory on the **upper half plane**, H. Then, since the conformal Killing fields, $\xi = \xi^z \partial_z + \bar{\xi}^{\bar{z}} \partial_{\bar{z}}$ should preserve the boundary, ie. the real line $\mathbb{R}$, we have $\xi^z(x) = \bar{\xi}^{\bar{z}}(x)$ for $x \in \mathbb{R}$ and thus, by analyticity, $\bar{\xi}^{\bar{z}}(\bar{z}) = \xi^z(z^*)$, where z^* is the reflection of z^* on the real line.
- This is consistent with the boundary condition $T_{zz}|_{\mathbb{R}} = \bar{T}_{\bar{z}\bar{z}}|_{\mathbb{R}}$ which, when translated into $T_{\tau\sigma}$ and after Wick-rotation into $T_{t\sigma}$ expresses the fact that **no momentum flows accross the boundary**. We will assume this boundary condition in what follows.
- For primary fields the OPE necessarily contains T_{zz} (or $\bar{T}_{\bar{z}\bar{z}}$). Thus, since $T_{zz}|_{\mathbb{R}} = \bar{T}_{\bar{z}\bar{z}}|_{\mathbb{R}}$ we infer from ($x \in \mathbb{R}$)
$$\phi_h(x) \times \phi_h(0) \sim T_{zz} = \bar{T}_{\bar{z}\bar{z}} \sim \bar{\phi}_h(x) \times \bar{\phi}_h(0)$$
that $\phi_h(x) = \pm\bar{\phi}_h(x)$ and thus by analyticity[1]
$$\bar{\phi}_h(\bar{z}) = \pm\phi_h(z^*)\,.$$
- **Conformal Ward Identity:** For
$$X := \phi_{h_1}(z_1)\bar{\phi}_{\bar{h}_1}(\bar{z}_1)\cdots\phi_{h_n}(z_n)\bar{\phi}_{\bar{h}_n}(\bar{z}_n) = \pm\phi_{h_1}(z_1)\phi_{\bar{h}_1}(z_1^*)\cdots\phi_{h_n}(z_n)\phi_{\bar{h}_n}(z_n^*)$$
the conformal Ward Identity on H can be reformulated as
$$\begin{aligned}\delta_{\xi\bar{\xi}}\langle X\rangle &= \frac{1}{2\pi i}\oint_C dz\;\xi(z)\langle T(z)X\rangle - \frac{1}{2\pi i}\oint_C d\bar{z}\;\bar{\xi}(\bar{z})\langle\bar{T}(\bar{z})X\rangle\\ &= \frac{1}{2\pi i}\oint_C dz\;\xi(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_{C^*} d\bar{z}\;\bar{\xi}(\bar{z})\langle\bar{T}(\bar{z})X\rangle\\ &= \pm\frac{1}{2\pi i}\oint_{C\cup C^*} dz\;\xi(z)\langle T(z)\phi_{h_1}(z_1)\phi_{\bar{h}_1}(z_1^*)\cdots\phi_{h_n}(z_n)\phi_{\bar{h}_n}(z_n^*)\rangle \qquad (23)\end{aligned}$$
which is a contour integral of a single holomorphic field on the full complex plane, $\mathbb{C}$.
- **one point functions:** The one point correlator on a CFT on H is given by
$$\langle\Phi_{h\bar{h}}(z,\bar{z})\rangle_H = \pm\langle\phi_h(z)\bar{\phi}_{\bar{h}}(z^*)\rangle_{\mathbb{C}} = \frac{c\;\delta_{h\bar{h}}}{z - z^*}$$
where c depends on the boundary condition.

23. Lecture on Jan. 22. – **Boundary Operators** [F]

- Boundary primary fields that are compatible with given boundary conditions ca be obtained by pulling bulk fields to the boundary. Using mirror fields we have
$$\lim_{z\to\mathbb{R}}\phi_{h\bar{h}}(z\bar{z}) = \lim_{z\o z^*}\phi_h(z)\phi_{\bar{h}}(\bar{z}) = \sum_i \frac{1}{(z-z^*)^{h+\bar{h}-h_i}}\phi_{h_i}(x)$$
where $x \in \mathbb{R}$ and $\phi_{h_i}(x)$ are **boundary fields**. For example for the Ising model we have
$$\sigma(z)\sigma(z^*) \sim a\mathbb{I} + b\epsilon(x)\,, \qquad \epsilon(z)\epsilon(z^*) \sim c\mathbb{I} \qquad (24)$$
where $a, c \neq 0$, and $b \neq 0$ for free boundary conditions.

[1] for multicomponent fields, more general boundary conditions are possible, see e.g. [RS].

- In addition there are boundary operators that cannot be obtained from bulk fields. These do change the boundary conditions. If $\phi_{ab}(x)$ is such a boundary operator it will change the boundary condition (a) to boundary condition (b) at the point x.
- In order to see which boundary operators can arise as a representation of the 'Virasoro algebra on computes the (finite temperature) partition function on the strip with boundary conditions (a) and (b) on each end:

$$Z_{ab} = \mathrm{Tr}_{H_{ab}}\left(e^{-\frac{\beta\pi}{L}(L_0-\frac{c}{24})}\right) = \sum_i n^i_{ab}\chi_i(q)$$

  where $\chi_i(q)$ is the character of the Verma module $\mathcal{V}_i$ for $q = e^{-\frac{\beta\pi}{L}}$ (see lecture 15) and n^i_{ab} is the number of times $\mathcal{V}_i$ appears.
- equivalently, this partition sum can be interpreted as the **bulk** CFT **transition amplitude** between **boundary states** $|a>$ and $|b>$,

$$Z_{ab} =< a|e^{-\frac{2\pi L}{\beta}(L_0+\bar{L}_0-\frac{c}{12})}|b> \tag{25}$$

- A basis for the consistent boundary states is given by the **Ishibashi states**

$$|B_{h_i}>>= \sum_N |h_i, N> \otimes \overline{|h_i, N>}$$

  where N labels the descendents.

24. Lecture on Jan. 24. – **Cardy Conditions and Verlinde Formula** [F]

- From our earlier definition of the characters of Verma modules it the follows that

$$Z_{ab} = \sum_j << h_j|a>< b|h_j>> \ \chi_j(e^{-\frac{4\pi L}{\beta}}) \tag{26}$$

- On the other hand, modular invariance implies

$$\chi_i(e^{-\frac{\pi\beta}{L}}) = \sum_j S_{ij}\chi_j(e^{-\frac{4\pi L}{\beta}}) \tag{27}$$

  where S_{ij} is symmetric and orthogonal.
- Equivalence of (25) and (26) then implies the **Cardy Conditions**

$$<< h_j|a>< b|h_j>>= \sum_i n^i_{ab}S_{ij}\ , \quad n^i_{ab} = \sum_j S_{ij} << h_j|a>< b|h_j>> \tag{28}$$

  where n^i_{ab} are integers.
- In order to solve these equations we make the following Ansatz for the boundary states $|\tilde{l}>$

$$|\tilde{l}>= \sum_j \frac{S_{lj}}{\sqrt{S_{0j}}}|h_j>> \tag{29}$$

  That this Ansatz is consitent follows form **Verlinde's Fusion Rules** which state that consistency on a torus (modular invariance) implies implies that

$$\frac{S_{kj}S_{lj}}{S_{0j}} = \sum_i N^i_{kl}S_{ij} \tag{30}$$

  holds for any CFT with N^i_{kl} natural numbers.

- **Ising Model:** From the bulk theory we infer the matrix

$$S_{ij} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \end{pmatrix}$$

Thus the consitent boundary states are given by

$$\begin{aligned} |\tilde{0}> &= \sqrt{\frac{1}{2}}|\tilde{0}>> + \sqrt{\frac{1}{2}}|\tilde{\epsilon}>> + \frac{1}{2^{1/4}}|\tilde{\epsilon}>> \\ |\tilde{\epsilon}> &= \sqrt{\frac{1}{2}}|\tilde{0}>> + \sqrt{\frac{1}{2}}|\tilde{\epsilon}>> - \frac{1}{2^{1/4}}|\tilde{\epsilon}>> \\ |\tilde{\sigma}> &= |\tilde{0}>> - |\tilde{\epsilon}>> \end{aligned}$$

They correspond to the boundary condition $+$,$-$ and $free$ respectively.

25. Lecture on Jan. 29. – **Perturbed Boundary Conformal Field Theory** [RS]

- The idea is to deform the boundary condition by *exponentiate* a boundary field ψ as

$$\begin{aligned} <\cdots>_{\lambda\psi} &= <\cdots e^{\lambda\int_{\mathbb{R}}\psi(x)dx}> \\ &= <\cdots(1+\lambda\int_{\mathbb{R}}\psi(x)dx + \frac{\lambda^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(x_1)\psi(x_2)dx_1dx_2 + \cdots)> \end{aligned} \tag{31}$$

- In what follows we will only consider deformations by **marginal boundary operators**, that is, operators of conformal weight one, for which the coupling λ is dimensionless.
- **Ordering:** The definition of the exponential in (31) display an ordering problem for double (and higher) integrals since $\psi_i(x_1)\psi_j(x_2) \neq \psi_j(x_2)\psi_i(x_1)$ in general.
- **Def:** $\psi_i(x_1)$ and $\psi_j(x_2)$ are said to be **mutually local** if $\psi_i(x_1)\psi_j(x_2) = \psi_j(x_2)\psi_i(x_1)$ and $\psi_i(x)$ is said to be **self local** if $\psi_i(x)$ is local w.r.t. itself.
- **Remark:** Locallity implies that the OPE $\psi_i(x_1)\psi_j(x_2)$ has a well defined analytic continuation from $\mathbb{R}$ into $\mathbb{C}$. In particular, for a conformal weight one, self-local boundary field ψ we have

$$\psi(z)\psi(0) = \frac{1}{z^2} + \text{ regular}$$

26. Lecture on Jan. 31. – **Exactly Marginal Deformations** [RS]

- If the perturbing field $\psi(x)$ is not self-local then the regularised integral

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\psi(x_1)\psi(x_2)dx_1dx_2$$

in (31) will be (logartihmically) scale dependent which induces a non-vanishing beta function for λ and thus the perturbed CFT fails to be conformal on the boundary (although it still is in the bulk).

- If the perturbing field $\psi(x)$ is self local, then $\beta_\lambda = 0$ at this order. Furthermore, self-locality implies that the 3-point function $\langle\psi(x_1)\psi(x_2)\psi(x_3)\rangle$ vanishes, which, in turn, implies that $\beta_\lambda = 0$ at all orders since higher point functions factorize into 2- and 3-point functions. Consequently, if $\psi(x)$ is self-local, then the deformation induced by ψ is **exactly marginal**.
- **Remark:** Self-locality, exact marginality and vanishing of the 3-point function $\langle\psi(x_1)\psi(x_2)\psi(x_3)\rangle$ are synonomous.
- For self-local perturbations the OPE $\psi_i(x_1)_j(x_2)$ has a well defines analytic continuation into the complex plane. Therefore we can regularise the perturbation integrals by replacing

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \psi(x_1)\cdots\psi(x_n)dx_1\cdots dx_n$$

by

$$\int_{\gamma_1} \cdots \int_{\gamma_n} \psi(z_1)\cdots\psi(z_n)dz_1\cdots dz_n$$

where γ_i is a curve in the upper half pane with imaginary part $\mathrm{Im}\gamma_i = i\epsilon$.

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