Analysis of the Angular Momentum Algebra

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1 Introduction

Algebraic methods are potent in quantum mechanics. The two most important applications are the algebraic solution of the harmonic oscillator and the analysis of the angular momentum eigenstates. There also exists an algebraic treatment of the Hydrogen atom, a review of which is presented in [1]. As we are familiar with the algebraic techniques for the oscillator, we will study the angular momentum in these notes.

We will first go over the most important notions of representation theory, as well as some fundamental results and go on to study the irreducible representations of SU(2) and then decompose tensor product representations into irreducible representations. The main references are [2] and [3]. Other good references are [4],[5] as well as [6].

2 Lie Groups and Representations

Lie groups play an important role in physics, as they always appear when we deal with continuous symmetries. The most important Lie groups are SO(3), the three-dimensional rotation group, $SU(2)$ the quantum mechanical angular momentum, the Lorentz group for relativistic systems and SU(N) for internal symmetries in relativistic quantum field theory, like SU(2) for the weak interaction and colour SU(3) for the strong interaction in the Standard Model of Particle Physics.

2.1 Lie Groups

A **Lie group** is a group G whose elements can be labelled in a continuous and smoothly differentiable way by a set of parameters θ_a, where $a = 1, \ldots, N$, with N the **dimension** of the group. Mathematically speaking, Lie groups are groups as well as differentiable manifolds. An example of such a Lie group is SO(3), where each element, an orthogonal 3×3 matrix O, is parametrised by three angles, the Euler angles. The group composition is matrix multiplication, and the neutral element is the identity matrix $\mathbb{1}$. The matrices depend on the three angles via trigonometric functions, hence smoothly.

We will mainly be interested in simply connected Lie groups, where each element $g(\theta)$ is connected to the identity element $\mathbb{1}$, and every closed loop can be contracted to a single point. This is the case e.g. for SU(2), but not for SO(3).

2.2 Representations

We are interested in the action of these groups on vector spaces, which we study by considering the **representations** of these groups. A representation is merely a homomorphism (a structure-preserving map) from the group G to the linear maps on a vector space V, called the **automorphisms** (isomorphisms on the same space) of V

$$\begin{aligned} D_R \colon G &\to \mathrm{Aut}(V). \\ g &\mapsto D_R(g) \end{aligned}$$

Each group element g is assigned to a linear operator $D_R(g)$. For this to be a homomorphism, these operators must be compatible with the group structure, i.e. the neutral element e and the group multiplication get carried over to the linear operators, in the sense that

$$\begin{aligned} D_R(e) &= \mathbb{1}, \\ D_R(g_1)D_R(g_2) &= D_R(g_1 g_2). \end{aligned}$$

The **dimension** of the representation is the dimension of the underlying vector space V, which is also called the **basis** or **module** of the representation. Often, we will sloppily use the word representation for describing both the map as well as the basis of the representation, i.e. the vector space.

A simple example of such a representation is the so-called **fundamental** representation of the matrix group $\mathrm{SU}(N)$, where each group element U, a unitary $N \times N$ matrix, is represented by precisely this matrix acting on $\mathbb{C}^N$, i.e.

$$U \mapsto D_F(U) = U.$$

The fundamental representation has N complex dimensions. For $\mathrm{SU}(2)$, this fundamental representation is the action on $\mathbb{C}^2$, a generic element U in this representation takes the form

$$U = \begin{pmatrix} \cos\theta \mathrm{e}^{\mathrm{i}\alpha} & \sin\theta \mathrm{e}^{\mathrm{i}\beta} \\ -\sin\theta \mathrm{e}^{-\mathrm{i}\beta} & \cos\theta \mathrm{e}^{-\mathrm{i}\alpha}, \end{pmatrix}$$

depending on one angle θ and two phases α, β, and acts on a vector $v \in \mathbb{C}^2$ via matrix-vector multiplication

$$Uv = \begin{pmatrix} \cos\theta \mathrm{e}^{\mathrm{i}\alpha} & \sin\theta \mathrm{e}^{\mathrm{i}\beta} \\ -\sin\theta \mathrm{e}^{-\mathrm{i}\beta} & \cos\theta \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Since representations act on vector spaces, we are allowed to choose convenient bases. If we have an invertible linear map S, we can perform a change of basis

$$D(g) \to D'(g) = S^{-1} D(g) S,$$

which will then also be a representation of G. In this case, we call the representations D and D' **equivalent**. An example of such an equivalence is computed explicitly in Sec. 5 for the $s = 1$ representation of $\mathrm{SU}(2)$ and the generators of rotations.

If there exist invariant subspaces $W \subset V$ such that the image $D_R(g)W \subseteq W$ for all g, we call the representation D **reducible**, otherwise **irreducible**. Furthermore, if we can find an equivalent representation D'_R such that all matrices $D'_R(g)$ take a block-diagonal form, we call the representation **completely reducible**. This block-diagonal form means that we can write the basis of the representation as a direct sum of irreducible representations.

Irreducible representations are the fundamental building blocks of representation theory, since they do not mix under transformation under the group. Hence, if we analyse the irreducible representations first, we can conclude many properties of general representations by decomposing them into the irreducible ones.

There is an important theorem for compact finite-dimensional Lie groups regarding completely reducible representations.

Theorem 1. *Every finite-dimensional representation of a compact Lie group is completely reducible, i.e. the direct sum of irreducible representations.*

Most cases we are interested in are finite-dimensional, and there we can always split any reducible representation into a direct sum of irreducible ones. In particular, this theorem will allow us to decompose the SU(2) representations without worrying about the complete reducibility of the representation. The decomposition will always be justified.

2.3 Lie Algebras

Since the Lie group depends on its parameter smoothly, we can consider the linear approximation of the group, by expanding an element close to the identity to first order in the parameter θ_a. We obtain

$$D(\delta\theta) = \mathbb{1} + \mathrm{i}\delta\theta_a T_a,$$

where we defined the **generator** T_a of the Lie group,

$$T_a = -\mathrm{i}\left.\frac{\mathrm{d}}{\mathrm{d}\theta_a}\right|_{\theta=0} D(a).$$

In this way, we can compute a set of generators for each representation D of the group. Hence, each group representation induces a representation of these generators. There are as many generators as the dimension N of the Lie group, since the local approximation is a N-dimensional vector space, and T^a can be thought of as the basis vectors spanning this space. We could now go on and lift the notion of generators to the level of Lie groups without referring to a certain representation.

To go back to the large group elements, we can **exponentiate** the generators, as we will show in the following.

First, consider two infinitesimal transformations $D(\delta\theta_1)$ and $D(\delta\theta_2)$. The successive application gives, to first order in $\delta\theta_i$,

$$\begin{aligned} D(\delta\theta_1)D(\delta\theta_2) &= (\mathbb{1} + \delta\theta_{1,a}T_a)(\mathbb{1} + \delta\theta_{2,a}T_a) \\ &= \mathbb{1} + (\delta\theta_1 + \delta\theta_2)_a T_a + \mathcal{O}(\delta^2). \end{aligned}$$

On the infinitesimal level, the group multiplication corresponds to addition of the infinitesimal parameters. Suppose now we want to obtain the transformation $D(\theta)$ by combination of infinitesimal transformations. We define an angle $\delta\theta = \frac{\theta}{n}$, which will be infinitesimal in the limit $n \to \infty$. Consecutive application of the infinitesimal transformation should then give rise to the transformation $D(\theta)$, so we obtain

$$D(\theta) = \lim_{n\to\infty} \left(\mathbb{1} + \mathrm{i}\frac{\theta_a}{n}T_a\right)^n = \mathrm{e}^{\mathrm{i}\theta_a T_a}.$$

We see that, starting from the generators of connected groups, we obtain the **exponential parametrisation** of the group, in the form

$$D(\theta) = \mathrm{e}^{\mathrm{i}\theta_a T_a}.$$

Additionally, the generators satisfy an **algebra**

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c,$$

where $[,]$ is the **Lie bracket** and f^{abc} are the **structure constants** of the group. This algebra, called the **Lie algebra**, is very powerful, since, for simply connected groups, it encodes the entire information of the Lie group.

Note that the form of the algebra, in particular, the structure constants, do not depend on the chosen representation. In the subsequent discussion, we will use the commutator in place of the Lie bracket for our analysis of SU(2). If we are interested in unitary representations, these generators can be taken to be Hermitian and by Stone's theorem even self-adjoint.

An example for such a Lie algebra is the angular momentum algebra. Consider the fundamental representation of the three generators S_x, S_y, S_z, given by

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the algebra

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk}S_k.$$

The **Casimir operator** is an operator that commutes with each representation of a generator, i.e. the Casimir C satisfies

$$[C, D(T^i)] = 0,$$

for all representations of generators T^i.

For the example of a Casimir operator in the case of SU(2) is the total angular momentum $\boldsymbol{S}^2$. It satisfies $[\boldsymbol{S}^2, S_i] = 0$ for all i.

There is an essential result regarding Casimir operators and irreducible representations, namely Schur's Lemma.

Theorem 2 (Schur's Lemma)**.** *If $[D(g), A] = 0$ for all $g \in G$, for a finite-dimensional irreducible representation D, then $A \sim \mathbb{1}$.*

Schur's Lemma tells us, roughly speaking, that the different irreducible representations can be labelled by eigenvalues of the Casimir operators, as these have to act proportionally to the identity on such an irreducible representation. We will see an explicit example of this when analysing SU(2).

We have to make an important remark about the connection between the Lie group and its algebra. We can think of the Lie algebra as the linear approximation of the group. This immediately raises a problem, if two groups look the same locally, but have very different global properties. An example we already encountered is SU(2) and SO(3). Both groups have 3 generators S_i, satisfying the algebra

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk} S_k.$$

Both groups agree infinitesimally, however, SO(3) is not simply connected.

A good way to visualise this is to consider the parametrisation of SO(3) in terms of the axis of rotation $\boldsymbol{n}$ and the angle $\theta \in [0, \pi)$ about which we rotate. The opposite axis then gives the interval $(\pi, 2\pi)$. This parametrisation corresponds to a solid sphere of radius π, and each point in this sphere $\boldsymbol{p} = r\hat{\boldsymbol{p}}$ corresponds to a rotation by angle $\theta = r$ about the axis $\hat{\boldsymbol{p}}$. We need to match antipodal points on this sphere, since rotations by π about $\boldsymbol{n}$ and by π about $-\boldsymbol{n}$ have the same effect, as they are related to the same group element. This now means that we can find closed paths in this sphere that cannot be contracted to a single point. Consider, for example, a path starting at a point $p = (\boldsymbol{n}, \theta)$ inside the sphere, reaching $p' = (\boldsymbol{n}, \pi)$. This point is matched to its antipodal point $p'' = (-\boldsymbol{n}, \pi)$, so we can close the loop by going back to p. This way we constructed a closed loop that cannot be contracted, since p' and p'' are always antipodal. Hence SO(3) is not simply connected.

This does not happen for SU(2), as this corresponds to the surface of the 3-sphere, which is simply connected. In fact, SU(2) is the universal covering group of SO(3). This crucial fact will appear later when we discuss the quantum mechanical implementation of symmetries.

Finally, let us mention the **tensor product** of representations. If we have two representations D_1 and D_2, we can form the tensor product $D_1 \otimes D_2$ as usual for finite-dimensional vector spaces. Since this is again a representation of G, we can then proceed to reduce it into irreducible representations. We will use this technique for the addition of angular momentum.

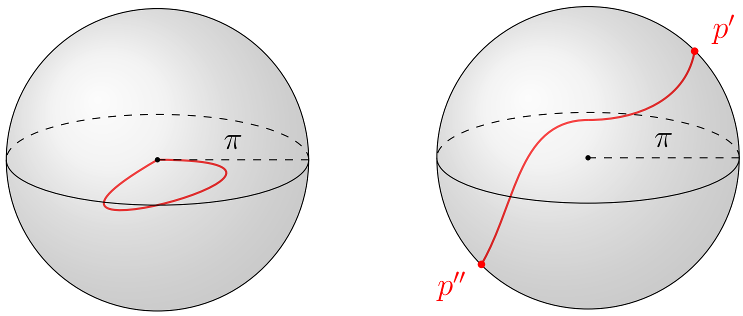

Figure 1: A picture of a contractible loop on the left-hand side, and a non-contractible loop on the right-hand side. Note that the points p' and p'' are identified.

3 Symmetries in Quantum Mechanics

Now, why are we interested in representations of symmetries in quantum mechanics? The short answer is that we can always take the eigenstates of the Hamiltonian to transform under some irreducible representation of the symmetry group G.

3.1 Symmetries

To begin, we have to clarify what we mean by a symmetry. First of all, a symmetry is a map $U\colon \psi \to U\psi$ that preserves the probabilistic interpretation of the theory, i.e.

$$|\langle U\psi \,|\, U\phi\rangle|^2 = |\langle \phi \,|\, \psi\rangle|^2 .$$

Such symmetries must be implemented quantum mechanically as **projective** transformations, meaning they do not affect probabilities. A transformation P is called projective if

$$P\,|\,\psi\rangle = e^{\mathrm{i}\phi}\,|\,\psi\rangle\,,$$

with $\phi \in \mathbb{R}$. A projective representation also has a modified multiplication identity, namely

$$P(g_1)P(g_2) = \mathrm{e}^{\mathrm{i}\phi(g_1,g_2)}P(g_1g_2),$$

again only respects the group structure up to a phase. This way, in a transition amplitude $\langle \phi \,|\, \psi\rangle$ we only obtain some phase factor, which cancels out when squaring it.

We cannot force the representations of a symmetry group to be unitary; it is enough for them to be projective in order to not destroy the probabilistic interpretation, due to the construction of the projective Hilbert space when considering rays instead of single states.

These projective representations can now be related bijectively to linear representations of **central extensions** of the symmetry group in question. For $SU(N)$, there are no non-trivial central extensions, so the projective one is equivalent to the unitary representation.

For $SO(N)$, however, there exist discrete central extensions, but these central extensions are the universal covering groups. Hence we can study projective representations of $SO(3)$ by considering unitary representations of the covering group $SU(2)$. Examples for a non-trivial central extension are, e.g. in CFT/String the Witt and Virasoro algebras.

With this knowledge, we expect in the quantum mechanical treatment of the angular momentum the appearance of SU(2) instead of SO(3). We know that the former is a double covering of the latter; hence we anticipate that this has some effect on the representation. It will turn out that this double covering manifests itself in the existence of half-integer spin representations.

In the following, we will always be considering situations where we can study the unitary representation of a symmetry group.

3.2 Dynamical Symmetries

This is, however, not the notion of symmetry we are familiar with from classical mechanics. Hence we need to introduce the notion of a **dynamical symmetry**. Recall that for classical systems, a symmetry maps one solution of the equations of motion into another solution. This symmetry was related to conserved charges via **Noether's theorem**. To obtain this notion in quantum mechanics, we need to consider the dynamics of the quantum system, in this case, the Schrödinger equation. We want the dynamical symmetries to map one solution of the Schrödinger equation to another one, i.e. we want for $|\,\psi\rangle$ satisfying

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\,\psi\rangle = H\,|\,\psi\rangle\,,$$

that $U\,|\,\psi\rangle$ is also a solution, namely

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}(U\,|\,\psi\rangle) = H(U\,|\,\psi\rangle).$$

If U is not explicitly time-dependent, we can move it past the derivative and obtain

$$\mathrm{i}\hbar U\frac{\mathrm{d}}{\mathrm{d}t}\,|\,\psi\rangle = UH\,|\,\psi\rangle = H(U\,|\,\psi\rangle).$$

We thus have to require $UH = HU$, or, using the commutator,

$$[U, H] = 0.$$

A dynamical symmetry is thus a unitary transformation that commutes with the Hamiltonian. Note that if $U = \mathrm{e}^{-\mathrm{i}\alpha T}$, this also implies

$$[T, H] = 0.$$

There is still a connection to conserved charges, similar to classical mechanics. The generator T is a Hermitian operator that commutes with the Hamiltonian, hence we can use a simultaneous eigenbasis of H and T. Let us expand a generic state $|\,\psi\rangle$ in this basis as

$$|\,\psi\rangle\,(t=0) = \sum_i c_i\,|\,E_i, \lambda_i\rangle\,,$$

where $H\,|\,E_i, \lambda_i\rangle = E_i\,|\,E_i, \lambda_i\rangle$ and $T\,|\,E_i, \lambda_i\rangle = \lambda_i\,|\,E_i, \lambda_i\rangle$. The time-evolution of these states is trivial since they are energy eigenstates. We find

$$|\,\psi\rangle\,(t) = \sum_i c_i \mathrm{e}^{-\mathrm{i}\frac{E_i}{\hbar}t}\,|\,E_i, \lambda_i\rangle\,.$$

However, this means that time-evolution does not mix states with different λ_i, i.e. the probability of measuring the eigenvalue λ_j is always

$$P(\lambda_j; t) = \left|c_j \mathrm{e}^{-\mathrm{i}\frac{E_j}{\hbar}t}\right|^2 = |c_j|^2\,,$$

independent of t. Hence, we conclude that the observable T is a **conserved charge**, in the sense that its eigenstates are not mixed under time-evolution. We can restate Noether's theorem for quantum mechanical systems in the following way: if a group G is a dynamical symmetry of the Hamiltonian H, meaning

$$[H, D_R(g)] = 0$$

for some representation D_R of G and all $g \in G$, or, equivalently,

$$[H, D(T)] = 0,$$

for some representation of the **generators** of G, then $D(T)$ is conserved observable.

Now let us return to the claim in the beginning, that each energy eigenstate transforms under some representation of such a dynamical symmetry.

To see this, consider a quantum mechanical system with a Hamiltonian H and a symmetry group given by G. Since G is a dynamical symmetry of H, we know that $[D(g), H] = 0$ for all g, so each eigenspace of H is the basis of a reducible representation of G. By theorem 1, we know that each representation can be completely reduced to irreducible representations. By studying irreducible representations, we can thus gain a better understanding of the systems and find good labels for the eigenstates.

4 Irreducible Representations of SU(2)

Let us now analyse the irreducible representations of SU(2), the group of unitary 2×2 matrices. It has three generators, which we define as $S_i = \frac{1}{2}\sigma_i$, where σ_i are the Pauli matrices. These generators satisfy the algebra

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk} S_k.$$

There is one Casimir operator, the **total angular momentum** $\boldsymbol{S^2} = S_x^2 + S_y^2 + S_z^2$, which commutes with the generators

$$[\boldsymbol{S}^2, S_i] = 0.$$

Suppose now that we consider an irreducible representation of SU(2), with states $|\,\lambda, m\rangle$, such that

$$\boldsymbol{S}^2 \,|\,\lambda, m\rangle = \lambda \,|\,\lambda, m\rangle \,,$$
$$S_z \,|\,\lambda, m\rangle = m \,|\,\lambda, m\rangle \,.$$

We will analyse this representation algebraically, by employing the so-called **ladder operators** $S_\pm$, defined as

$$S_+ = S_x \pm \mathrm{i} S_y.$$

Quantum mechanically, this means that $\boldsymbol{S}^2$ and S_z are our observables, whereas $S_\pm$, failing to be Hermitian, are merely tools we employ to study the eigenvalues of these observables. These operators satisfy the commutation relations

$$[S_+, S_-] = 2S_z$$
$$[\boldsymbol{S}^2, S_\pm] = 0$$
$$[S_z, S_\pm] = \pm S_\pm.$$

This means that they do not change the $\boldsymbol{S}^2$ eigenvalue, but they do change the S_z eigenvalue. The name "ladder operator" arises from their action on S_z eigenstates, since

$$S_z \left(S_\pm \,|\,\lambda, m\rangle\right) = [S_z, S_\pm] \,|\,\lambda, m\rangle + S_\pm S_z \,|\,\lambda, m\rangle = (m \pm 1) \left(S_\pm \,|\,\lambda, m\rangle\right),$$

in other words

$$S_\pm \,|\,\lambda, m\rangle = \alpha^\pm_{\lambda, m} \,|\,\lambda, m \pm 1\rangle \,,$$

so the operator S_+ raises the S_z eigenvalue by one unit, and S_- lowers it. We will worry about the normalisation later. Note that this means in particular, that we never leave the irreducible representation by applying $S_\pm$, as it is labelled by the $\boldsymbol{S}^2$ eigenvalue.

Next, note that in each irreducible representation, this ladder of S_z eigenvalues is finite. We can see this by considering

$$0 \leq \langle \lambda, m \,|\, S_x^2 + S_y^2 \,|\, \lambda, m \rangle = \left\langle \lambda, m \,\middle|\, \boldsymbol{S}^2 - S_z^2 \,\middle|\, \lambda, m \right\rangle ,$$

i.e. $m^2 \leq \lambda$. With this result, we can immediately conclude that there exists a maximal m, which we call s, and which satisfies

$$S_+ \,|\, \lambda, s \rangle = 0,$$

else s would not be the maximal m. The second option is that $S_+ \,|\, \lambda, s\rangle$ is not normalisable, but this case would be unphysical hence we discard it.

Applying the ladder to this state, we will eventually land at some minimal state with $m = s'$, which satisfies

$$S_- \,|\, \lambda, s' \rangle = 0.$$

From this we can already conclude that $s - s' \in \mathbb{N}$, since we obtain s' by applying S_- some finite number of times.

Next, we will use these states to fix λ and s' in terms of s. We use

$$\begin{aligned} 0 &= S_- S_+ \,|\, \lambda, s \rangle = (\boldsymbol{S}^2 - S_z^2 - S_z) \,|\, \lambda, s \rangle = (\lambda - s^2 - s) \,|\, \lambda, s \rangle , \\ 0 &= S_+ S_- \,|\, \lambda, s' \rangle = (\boldsymbol{S}^2 - S_z^2 + S_z) \,|\, \lambda, s' \rangle = (\lambda - (s')^2 + s') \,|\, \lambda, s' \rangle , \end{aligned}$$

and compare both values for λ to find

$$s(s+1) = s'(s'-1).$$

This is a quadratic equation with the solutions $s' = -s$ and $s' = s + 1$. Since s is the maximal eigenvalue, the second solution cannot hold, so we conclude $s' = -s$. Note that since $s - s' \in \mathbb{N}$, we conclude that $2s \in \mathbb{N}$. We now choose to label λ by s, i.e. we have states

$$\begin{aligned} \boldsymbol{S}^2 \,|\, s, m \rangle &= s(s+1) \,|\, s, m \rangle , \\ S_z \,|\, s, m \rangle &= m \,|\, s, m \rangle . \end{aligned}$$

We can now say more about the irreducible representations of SU(2). Each such representation is labelled by the eigenvalue s of the Casimir operator, $\boldsymbol{S}^2$, and the S_z eigenvalue m. This m can take values in $\{-s, -s+1, \ldots, s-1, s\}$, i.e. each representation is $2s+1$-dimensional. Finally, s can either be integer or half-integer.

We can best visualise this analysis by thinking of the states $|\, s, m\rangle$ forming a ladder with $2s + 1$ rungs that are labelled by the corresponding values of m. The operators $S_\pm$ move from the rung corresponding to m to the one directly above/below.

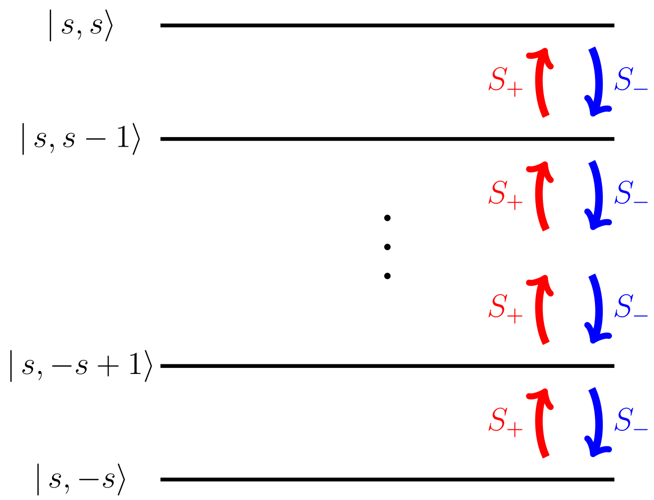

Figure 2: A visualisation of the ladder of S_z eigenstates $|\, s, m\rangle$ and the action of the ladder operators.

Let us also take a look at the proper normalisation of the states. If we define the states $|\, s, m\rangle$ to be orthonormal, we have

$$S_\pm \,|\, s, m\rangle = \alpha^\pm_{sm} \,|\, s, m \pm 1\rangle \,.$$

To compute these coefficients, we consider

$$\begin{aligned} \left|\alpha^\pm_{sm}\right|^2 &= \langle s, m \pm 1 \,|\, s, m \pm 1\rangle \\ &= \langle s, m \,|\, S_\mp S_\pm \,|\, s, m\rangle \\ &= \left\langle s, m \,\middle|\, \boldsymbol{S}^2 - S_z^2 \mp S_z \,\middle|\, s, m\right\rangle \\ &= \left\langle s, m \,\middle|\, s(s+1) - m^2 \mp m \,\middle|\, s, m\right\rangle \\ &= s(s+1) - m(m \mp 1). \end{aligned}$$

These coefficients correspond to $N_m = \sqrt{(s+m)(s-m+1)}$ from the problem sets, in the convention

$$S_- \,|\, s, m\rangle = N_m \,|\, s, m-1\rangle \,, \quad S_+ \,|\, s, m\rangle = N_{m+1} \,|\, s, m+1\rangle \,.$$

5 Example Representations

It is instructive to consider some examples. We will take a look at the $s = \frac{1}{2}$ and $s = 1$ representation of SU(2). For $s = \frac{1}{2}$, we fix the basis

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

and we can obtain S_x and S_y from the ladder operators by inverting the formulas. We have

$$\begin{aligned} S_x &= \frac{1}{2} \left(S_+ + S_- \right) , \\ S_y &= -\frac{\mathrm{i}}{2} \left(S_+ - S_- \right) . \end{aligned}$$

To find the explicit form, we use the algebra, namely

$$\begin{aligned} S_+ \left| s, m \right\rangle &= N_{m+1} \left| s, m+1 \right\rangle , \\ S_- \left| s, m \right\rangle &= N_m \left| s, m-1 \right\rangle , \\ S_z \left| s, m \right\rangle &= m \left| s, m \right\rangle . \end{aligned}$$

The corresponding matrix representations are thus given by

$$S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

and thus

$$S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad S_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S_y = \frac{1}{2} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} .$$

For $s = 1$, we use

$$\left| 1, 1 \right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \left| 1, 0 \right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \left| 1, -1 \right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

and find

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} .$$

This looks somewhat unfamiliar. It is instructive to see that this representation is, in fact, equivalent to the known one from classical mechanics. From the previous problem sets, we constructed the generators of rotation as

$$S_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} .$$

These matrices are anti-symmetric, hence the Hermitian generators are seen to be

$$S_z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}.$$

We would expect this to be an equivalent representation. Indeed, if we diagonalise the S_z matrix in this basis, via the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} & 0 & \mathrm{i} \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0, \end{pmatrix}$$

we find

$$U^\dagger S_z U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The other generators of the first set are similarly obtained using this matrix U. As expected, the constructed representation and the one we derived are equivalent.

From this explicit representation of the generators, we can now go on and find rotation matrices by exponentiation.

6 Tensor Product: $\frac{1}{2} \otimes 1$

Next, let us move on to tensor products of representations, which appear quite naturally in quantum mechanics, e.g. systems with orbital angular momentum and spin, or multiple particles with spin. The decomposition of these representations into irreducibles will be similar to the procedure above.

First, we will find the irreducible representation with the maximal total angular momentum j, then construct the ladder, and move on to the next irreducible representation with $j-1$. We will continue this procedure until we "run out of space", i.e. until we have decomposed the representation.

First, we have to organise the ladder operators. Since they are constructed from generators, we should take a look at the generators in the product representation. Consider two representations $D_1(g)$ and $D_2(g)$. Then their tensor product acts as

$$\begin{aligned} D_{1\otimes 2}(g)\,|\,i,x\rangle &= |\,j,y\rangle\,[D_{1\otimes 2}(g)]_{jy\,ix} \\ &= |\,j\rangle\,|\,y\rangle\,[D_1(g)]_{ji}[D_2(g)]_{yx} \\ &= (|\,j\rangle\,[D_1(g)]_{ji})(|\,y\rangle\,[D_2(g)]_{xy}), \end{aligned}$$

where we used the label $i = s_1, m_1$ and $x = s_2, m_2$ as short-hand notation. Expanding this to first order in the parameter α, we find

$$(1 + \mathrm{i}\alpha_a S_{a,1\otimes 2})\,|\,i,x\rangle = |\,j,y\rangle\,(\delta_{ji} + \mathrm{i}\alpha_a[S_{a,1}]_{ji})(\delta_{yx} + \mathrm{i}\alpha_a[S_{a,2}]_{yx}.$$

We see that the generator of the product transformation is simply the "sum" of the generators, in the sense $S \otimes \mathbb{1} + \mathbb{1} \otimes S$.

This important result allows us to make two conclusions. First, the S_z values add for tensor representations, and second, we can construct ladder operators by "adding" the ladder operators of the original spaces in the same way as the generators. Instead of showing the general procedure, we will analyse the simple case $s_1 = \frac{1}{2}$ and $s_2 = 1$, i.e. the product representation $\mathbf{\frac{1}{2}}\otimes\mathbf{1}$. We want to decompose this representation in irreducible representations with some total angular momentum j, i.e. into states $|\,j, m_j\rangle$.

Since the values for m add, we see that the maximal m_j is given by $m_j = \frac{3}{2}$, with corresponding $j = \frac{3}{2}$. There is only one state in the tensor product that gives $m = \frac{3}{2}$, namely $\left|\,\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |\,1,1\rangle$. We thus find the first state

$$\left|\,\frac{3}{2}, \frac{3}{2}\right\rangle = \left|\,\frac{1}{2}, \frac{1}{2}\right\rangle \otimes |\,1,1\rangle\,.$$

Next, we act with the ladder operator S_- on this state. We use

$$\begin{aligned} S_+ &= S_{+,\mathbf{\frac{1}{2}}} \otimes \mathbb{1}_{\mathbf{1}} + \mathbb{1}_{\mathbf{\frac{1}{2}}} \otimes S_{+,\mathbf{1}}, \\ S_- &= S_{-,\mathbf{\frac{1}{2}}} \otimes \mathbb{1}_{\mathbf{1}} + \mathbb{1}_{\mathbf{\frac{1}{2}}} \otimes S_{-,\mathbf{1}}. \end{aligned}$$

and compute, keeping in mind the normalisation N_m,

$$S_- \left| \frac{3}{2}, \frac{3}{2} \right\rangle = \sqrt{3} \left| \frac{3}{2}, \frac{1}{2} \right\rangle ,$$

and on the left-hand side

$$\begin{aligned} S_- \left| \frac{3}{2}, \frac{3}{2} \right\rangle &= (S_- \left| \frac{1}{2}, \frac{1}{2} \right\rangle) \left| 1, 1 \right\rangle + \left| \frac{1}{2}, \frac{1}{2} \right\rangle (S_- \left| 1, 1 \right\rangle) \\ &= \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, 1 \right\rangle + \sqrt{2} \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, 0 \right\rangle . \end{aligned}$$

We continue this ladder to find the other two states. In the end, we have four states

$$\begin{aligned} \left| \frac{3}{2}, \frac{3}{2} \right\rangle &= \left| \frac{1}{2}, \frac{1}{2} \right\rangle \otimes \left| 1, 1 \right\rangle , \\ \left| \frac{3}{2}, \frac{1}{2} \right\rangle &= \frac{1}{\sqrt{3}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, 1 \right\rangle + \sqrt{\frac{2}{3}} \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, 0 \right\rangle , \\ \left| \frac{3}{2}, -\frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, 0 \right\rangle + \frac{1}{\sqrt{3}} \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, -1 \right\rangle , \\ \left| \frac{3}{2}, -\frac{3}{2} \right\rangle &= \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, -1 \right\rangle . \end{aligned}$$

This is the complete $j = \frac{3}{2}$ representation. The tensor product, however, is 6 dimensional. Hence there must be two states forming the $j = \frac{1}{2}$ representation, which are orthogonal to these states, since they have a different j eigenvalue.

The easiest way is to consider the $m = \frac{1}{2}$ states and find the orthogonal combinations. We obtain

$$\begin{aligned} \left| \frac{1}{2}, \frac{1}{2} \right\rangle &= \sqrt{\frac{2}{3}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, 1 \right\rangle - \sqrt{\frac{1}{3}} \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, 0 \right\rangle \\ \left| \frac{1}{2}, -\frac{1}{2} \right\rangle &= \sqrt{\frac{1}{3}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| 1, 0 \right\rangle - \sqrt{\frac{2}{3}} \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| 1, -1 \right\rangle , \end{aligned}$$

which are in fact related by the ladder operator S_-. We thus found the six states spanning the product space, and conclude that the tensor representation can be decomposed into

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \mathbf{1} = \frac{\mathbf{3}}{\mathbf{2}} \oplus \frac{\mathbf{1}}{\mathbf{2}}.$$

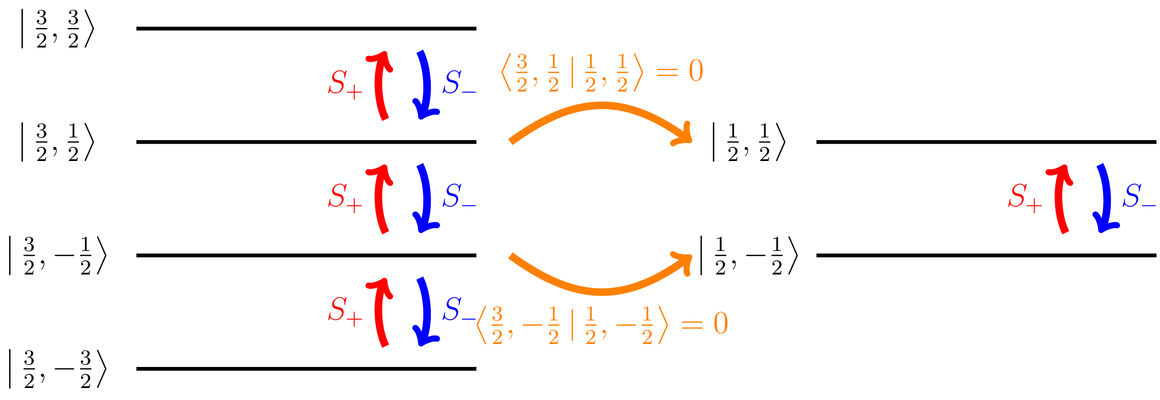

Figure 3: A visualisation of the procedure of finding the decomposition into the irreducible representations for $\mathbf{\frac{1}{2}} \otimes \mathbf{1}$.

7 The General Case

For a general product, the procedure is analogous to the previous discussion. First, we compute the maximal m value by adding the maximal eigenvalues of all factors. Next, we construct the corresponding spin-m-representation by acting with S_- on the maximal state. Once this process terminates, we consider the next-lower spin-$m-1$-representation and do the same analysis. We perform this procedure until we have obtained $\mathbf{s}_1 \cdot \mathbf{s}_2$ orthogonal vectors. The claim, based on our example above, is that this continues until the lowest spin $|s_1 - s_2|$. We show this inductively. The claim is

$$(2s_1+1)(2s_2+1) = \sum_{j=|s_1-s_2|}^{s_1+s_2} (2j+1).$$

As base step, consider $s_1 = s_2 = \frac{1}{2}$. We have

$$2 \cdot 2 = 3 + 1.$$

Now suppose $s_1 \geq s_2$. We first perform induction in s_1, i.e. $s_1 \to s_1 + \frac{1}{2}$. We have

$$(2(s_1+\frac{1}{2})+1)\cdot(2s_2+1) = (2s_1+1)(2s_2+1) + 2s_2 + 1 = \sum_{j=s_1-s_2}^{s_1+s_2}(2j+1) + 2s_2 + 1.$$

Note that the sum contains $2s_2+1$ terms, so we use the $2s_2+1$ to add one to each term, resulting in

$$\sum_{j=s_1-s_2}^{s_1+s_2}(2j+1) + 2s_2 + 1 = \sum_{j=s_1-s_2}^{s_1+s_2}(2(j+\frac{1}{2})+1) = \sum_{j=(s_1+\frac{1}{2})-s_2}^{(s_1+\frac{1}{2})+s_2}(2j+1).$$

Induction in s_2 works analogous, we have to watch out for the case where $s_2 + \frac{1}{2} > s_1$, this is where the absolute value comes in. We now know that the dimensions match, and we know how to find the complete set of states in each spin-ℓ-representation, so we conclude that indeed

$$\mathbf{s}_1 \otimes \mathbf{s}_2 = \bigoplus_{j=|s_1-s_2|}^{s_1+s_2} \mathbf{j}.$$

8 Conclusion

In these notes, we gave a quick introduction into representation theory to motivate the analysis of the angular momentum algebra. Based on the algebraic method using ladder operators, we studied the irreducible representations of SU(2). We then employed these techniques to decompose tensor products, first $\mathbf{\frac{1}{2}} \otimes \mathbf{1}$, then we decomposed a general tensor product, deriving the complete decomposition into irreducible representations for the product of two spin-s-representations.

While we were not very rigorous and somewhat handwavy in parts, this analysis can be performed in a similar fashion absolutely rigorous. This procedure, using so-called heighest-weight states, can be generalised to all compact Lie algebras and we use similar techniques to study internal symmetry groups or the Lorentz group for relativistic systems. The importance of these tools cannot be stressed enough, and will be important throughout the entire journey of learning theoretical physics.

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