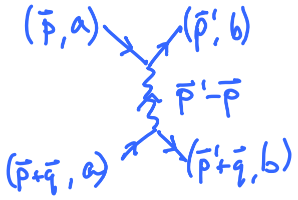
Infinite-order expansions

To illustrate idea, let $N \to \infty$ be control parameter in ϕ^4-theory:

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial\phi \cdot \partial\phi + \frac{r}{2} \phi \cdot \phi + \frac{g}{N} (\phi \cdot \phi)^2 \right) \qquad (1)$$

$\frac{g}{N}$ ← introduced for later convenience

with vector field: $\phi \equiv \{\phi^a\}$, $a = 1, \dots, N$ (2)



$$(\phi \cdot \phi)^2 = \sum_{ab} \phi^a \phi^a \phi^b \phi^b$$ (note: lines not all equivalent!) (3)

(no ω-momenta $p = \vec{p}, \omega_n$, since ϕ^4 has no dynamics, $\dot{\phi}^2$ is absent)

Goal: calculate $\mathcal{G}^{ab}(\vec{x}, \vec{y}) = \langle \phi^a(\vec{x})\, \phi^b(\vec{y}) \rangle$ (4a)

free GF: $\mathcal{G}_0 = \langle \qquad \rangle_0 \sim \delta^{ab}$ (4b)

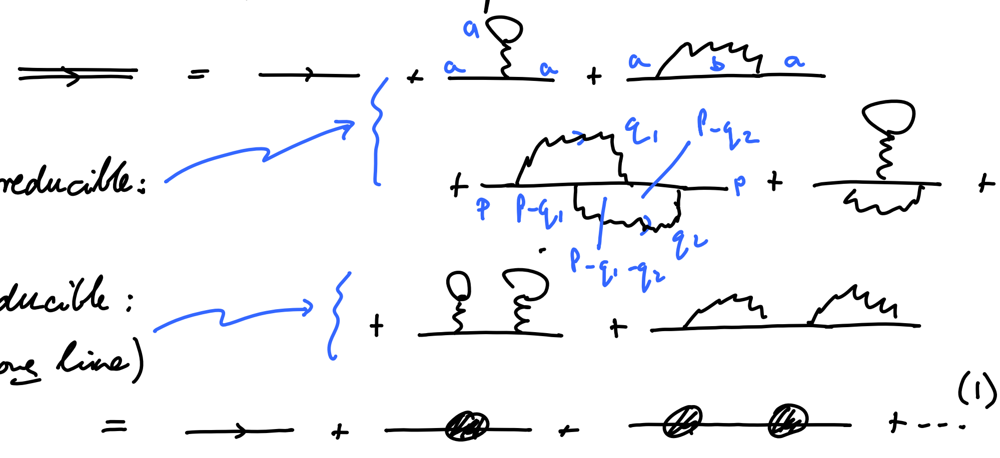
In other contexts, "large N" is spin S, or N_c = # of colors in QCD, or # of dimensions d (even though in practice $S = 1/2$, $N_c = 3$, $d = 3$ ... ☺)

Self-energy operator (or "effective mass operator")

Expand $\mathcal{G}$:

one-particle-irreducible:

one-particle-reducible: (splits by cutting one line)



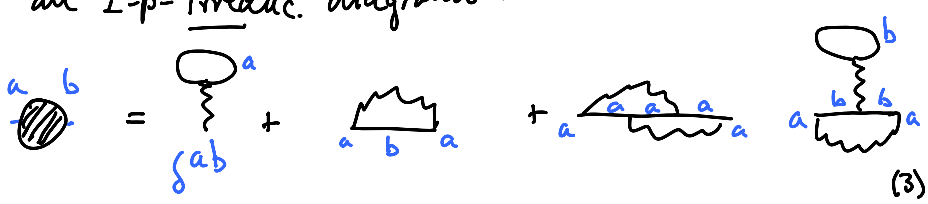
(1)

Dyson equation (DE):

(iterate to recover ↑) (2)

self-energy: collects all 1-p-irreduc. diagrams:

$\hat{\Sigma}^{ab}(\vec{x}-\vec{y}) = \hat{\Sigma} =$



(3)

PT21

Dyson eq. algebraically: $\hat{G} = \hat{G}_0 + \hat{G}_0 \hat{\Sigma} \hat{G}_0 + \hat{G}_0 \hat{\Sigma} \hat{G}_0 \Sigma \hat{G}_0$ (1)

$= \hat{G}_0 + \hat{G}_0 \hat{\Sigma} \hat{G}$ (2)

Operator product means: $(\hat{A}\hat{B})^{ab}(\vec{x},\vec{y}) = \int d^d z \; A^{ac}(\vec{x}-\vec{z})\, B^{cb}(\vec{z}-\vec{x})$ (3)

Fourier transform: $\hat{G}_{\vec{p}} = \hat{G}_{0,\vec{p}} + \hat{G}_{0,p}\, \hat{\Sigma}_p\, \hat{G}_p$ (4)

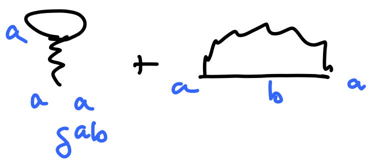
$\vec{p}$ is good quantum number, since conserved at each vertex!

solve for $\hat{G}$: $G_{\vec{p}} = \left[\hat{G}_0^{-1} - \hat{\Sigma}\right]^{-1} = (p^2 + r - \hat{\Sigma}_p)^{-1}$ (5)

$\hat{\Sigma}$ acts like a mass!

PT22

Calculate Σ order by order: $(\Sigma^{(1)})^{ab} =$



(1)

1st order: $\left[\hat{\Sigma}^{(1)}\right]^{ab}_{\vec{p}} = -\delta^{ab}\frac{2g}{L^d}\frac{1}{N}\left(\sum_{\vec{p}'} G_{0\vec{p}'} + N \sum_{p'} G_{0,\vec{p}-\vec{p}'}\right)$ (2)

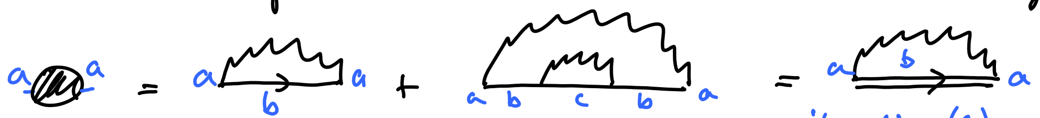
summation over b

Homework: check combinatorics of iterating DE!

<u>Large-N expansion:</u> find $\lim\limits_{N\to\infty} G_p^{aa}$: keep $O(1)$, neglect $O(\frac{1}{N})$:

- Each $\frac{1}{N}$ in vertex must be compensated by one $\sum_b \sim N$.

Only "rainbow" diagrams survive: (no bubbles, no crossing int. lines)



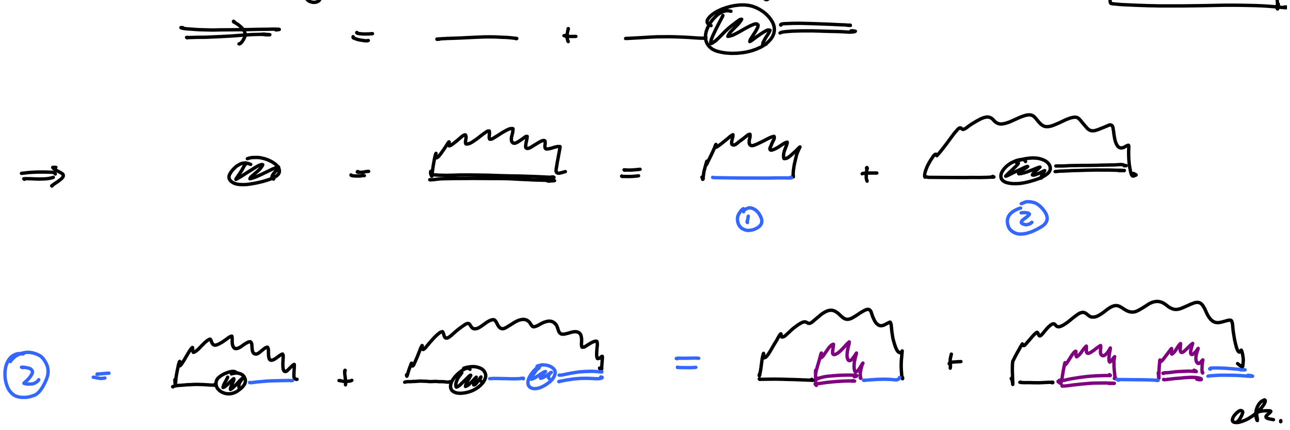
(3)

iteration (6) reproduces the series on LHS.

"Non-crossing-approximation"

PT23

Check: (6) & (15) generate all rainbow diagrams:



$$\text{(22.3)}\quad \left(\hat{\Sigma}_{\vec p}\right)^{ab\ \mathrm{NCA}} = -\delta^{ab}\frac{g}{L^d}\sum_{\vec p} G_{\vec p} = -\delta^{ab}\frac{g}{L^d}\sum_{\vec p}\left(p^2+r-\Sigma_p\right)^{-1} \qquad (1)$$

use full $\hat{\Sigma}_p$

1st order in g

This is called "self-consistent Born-approximation"

PT24

Example: evaluate $\hat{\Sigma}_p$ for $d=2$:

Need to introduce upper cut-off, $|p| < \Lambda$:

(Check!!) (1)

Then: $$\Sigma_p = \Sigma \sim -\frac{g}{4\pi}\int_0^{\Lambda^2} d(p)^2\,\frac{1}{p^2-\Sigma} \simeq -\frac{g}{4\pi}\ln\left(-\frac{\Lambda}{\Sigma}\right)$$

Ansatz: neglect p-dependence, check this self-consistently

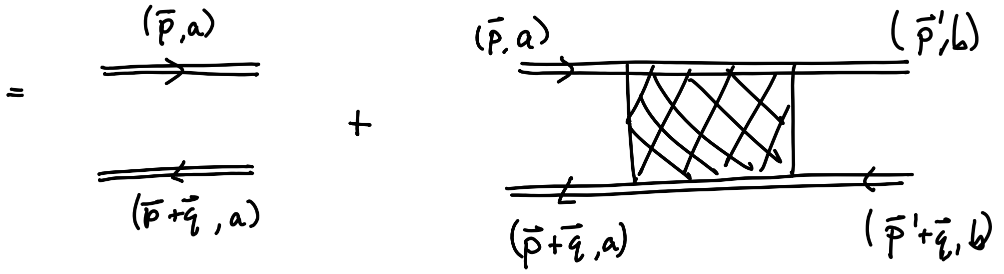
Note: interactions increase r to $r\underbrace{-\Sigma_p}_{>0,\ \text{see (16)}}$, i.e. reduce correlation length $\xi \sim r^{-2}$

makes sense: they lead to more scattering!

Why did pert. theory not diverge here? Because $N\to\infty$ killed infinitely many diagrams, they did not proliferate ...!

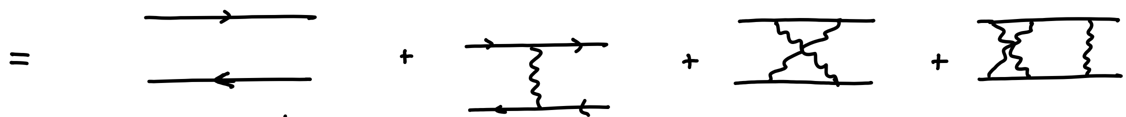
4-point functions:

$$C^{(4)}_{q} = \frac{1}{N}\sum_{ab}\frac{1}{L^{2d}}\sum_{pp'}\langle \phi^a_p\,\phi^a_{p+q}\,\phi^b_{p'}\,\phi^b_{p'+q}\rangle \qquad (1)$$



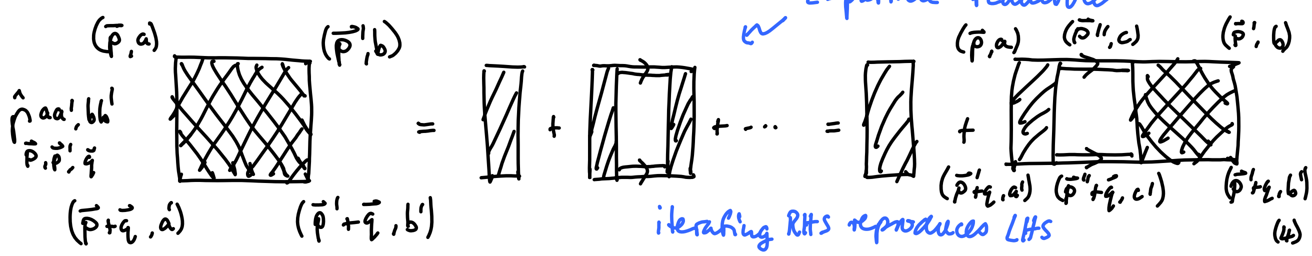
(2)

(3)



removing legs gives "vertex":

"2-particle-reducible"

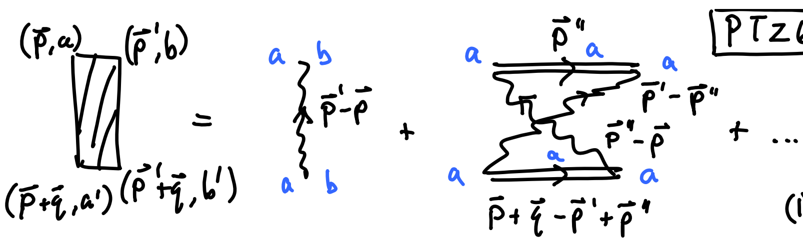


$\hat{\Gamma}^{aa',bb'}_{\vec p,\vec p',\vec q}$

iterating RHS reproduces LHS

(4)

"irreducible vertex" $\hat{\Gamma}^{aa',bb'}_{0\,\vec p,\vec p',\vec q}$



(1)

(25.4) is graphical form of "Bethe-Salpeter-equation" (BSE):

$$\hat{\Gamma}^{aa',bb'}_{\vec p,\vec p',\vec q} = \hat{\Gamma}^{aa',bb'}_{0\,\vec p,\vec p',\vec q} + \frac{1}{L^d}\sum_{\vec p''}\hat{\Gamma}^{aa',cc'}_{0\,\vec p,\vec p'',\vec q}\,G^{c}_{\vec p''}\,G^{c'}_{\vec p''+\vec q}\,\hat{\Gamma}^{cc',bb'}_{\vec p,\vec p',\vec q} \qquad (2)$$

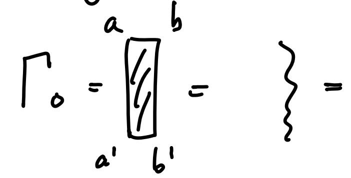
Note: massively self-consistent, highly non-linear!

Note: BSE and Dyson eq. are similar; general structure:

$$\hat{X} = \hat{X}_0 + \hat{X}_0 * \hat{\Sigma} * \hat{X} \qquad (3)$$

"full" "free" "irreducible" generalized matrix convolution

In large-N limit, only "non-entangled" contributions survive! $\boxed{\text{PT27}}$

$\Gamma_0 =$  $= -\frac{g}{N}\frac{1}{L^d}\,\delta^{aa'}\delta^{bb'}$ + terms of $O(g/N)$ smaller. (1)

(25.4): $\Rightarrow$ Γ  (2)

ladder!

(26.2): $$\Gamma^{ab}_{\vec p,\vec p',\vec q} = -\frac{g}{NL^d} - \frac{g}{NL^d}\sum_c\sum_{\vec p''}\underbrace{G^c_{\vec p''}\,G^c_{\vec p''+\vec q}}_{P_{\vec q}}\,\Gamma^{cb}_{\vec p'',\vec p,\vec q} \quad (3)$$

Try Ansatz: $\overset{|||}{\Gamma_{\vec q}}$ (no $\vec p, \vec p'$, a,b-dependence, striving for maximum simplicity) (4)

$$\Gamma_{\vec q} = -\frac{g}{NL^d} - g\,P_{\vec q}\,\Gamma_{\vec q} \quad \Rightarrow \quad \Gamma_{\vec q} = -\frac{g}{NL^d}\,\frac{1}{1+P_q g}$$

In principle, now calculate P_q using $(12)_{N\to\infty}$ for $G_{\vec p}$, $\boxed{\text{PT28}}$

to find Γ and $\quad C^{(4)} \overset{(19)}{=} P_q\left[\frac{1}{L^d} + N\Gamma_q P_q\right]$, etc. ... (1)

We'll do this for more interesting problems than ϕ^4 later!

<u>Comment:</u> Consider expansion $P_q = P(0) + \vec q^{\,2} P'(0) + \dots$ (2)

$$P(0) = \frac{1}{L^d}\sum_{\vec p} G^2_{\vec p} \overset{(12)}{=} \partial_{\Sigma_p}\underbrace{\frac{1}{L^d}\sum_{\vec p} G_{\vec p}}_{(16)} \qquad \text{since } G_p = \frac{1}{G^{-1}_{0,p}-\Sigma_p}$$

$$= \partial_{\Sigma_p}\left(-\frac{1}{g}\Sigma_{\vec p}\right) = -\frac{1}{g} \quad (3)$$

(3) in (27.4) $\Rightarrow$ $\Gamma_{\vec q} = -\frac{g}{NL^d}\,P'(0)\,\vec q^{\,2} \longrightarrow 0$ as $\vec q \to 0$ (4)

$\Rightarrow$ $\Gamma(\vec r) \sim |r|^{-\nu}$ decays as power law (as does $C^{(4)}$), <u>not</u> exponentially

$\Rightarrow$ vertex and 4-point-functions are long-ranged!!

General comments:

- summable classes of diagrams: ring-, rainbow-, bubble-diagrams (not many more classes are known!)
- Important structural elements (subunits) of diagrams (like Σ, Π, P) are constructed from <u>bare</u> quantities, G_0, V_0, etc.:

$\Sigma =$ , $\Pi =$ , ...

- Would be more convenient/conceptually nicer to reformulate theory such that the important structural elements directly represent the <u>elementary</u> degrees of freedom of theory

$\Rightarrow$ use functional integral to achieve this!!