Goldstone mode: Consequences of local U(1) gauge invariance

BCS 14

$$S_{EM} = \int d\tau d^d r \, \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

$$S[\bar\psi,\psi] = \int_0^\beta d\tau \int d^d r \Big\{ \bar\psi_\sigma(\vec r)\Big[\partial_\tau + ie\phi + \tfrac{1}{2m}(-i\vec\nabla - e\vec A)^2 - \mu\Big]\psi_\sigma(\vec r) - g\, c^\dagger_\uparrow(\vec r)\, c^\dagger_\downarrow(\vec r)\, c_\downarrow(\vec r)\, c_\uparrow(\vec r)\Big\} \tag{2}$$

S has local gauge invariance under:

$$\psi \to e^{i\theta}\psi, \quad \bar\psi \to e^{-i\theta}\bar\psi, \quad \phi \to \phi - \frac{\partial_\tau\theta}{e}, \quad \vec A \to \vec A + \frac{\vec\nabla\theta}{e} \tag{3}$$

After HS, introducing $\int\mathcal{D}(\Delta,\bar\Delta)$ and integrating out ψ:

$$Z = \int \mathcal{D}(\bar\Delta,\Delta)\, \exp\Big\{-\frac{1}{g}\int d\tau d^d r\, |\Delta|^2 + \ln\det \hat{\mathcal{G}}^{-1}\Big\}, \tag{4}$$

where
$$\hat{\mathcal{G}}^{-1} = \begin{bmatrix} \mathcal{G}^{(p)-1} & \Delta \\ \bar\Delta & \mathcal{G}^{(h)-1} \end{bmatrix} \tag{5}$$

with
$$\Big[\hat{\mathcal{G}}_0^{(p/h)}\Big]^{-1} = -\partial_\tau \mp ie\phi \mp \frac{1}{2m}(\mp i\vec\nabla - e\vec A)^2 \pm \mu \tag{6}$$

BCS 15

Question: what is the form of effective action describing <u>phase</u> fluctuations in order parameter field $\Delta(\vec r,\tau) = e^{2i\theta(\vec r,\tau)}\Delta_0$? (1)

Make informed guess, based on following arguments:

1) θ is Goldstone mode, θ-dependent terms in action must vanish if θ = const. $\Rightarrow$ $S = S(\nabla\theta, \partial_\tau\theta)$ (2)

2) assume $\nabla\theta$, $\partial_\tau\theta$ and ϕ, $\vec A$ are small, retain to lowest order only

3) symmetry: only even powers of $\nabla\theta$, $\partial_\tau\theta$, and no mixed terms: $\cancel{\partial_\tau\theta\,\vec\nabla\theta}$

4) invariance under local gauge transf.,
$$\begin{cases} \theta(\vec r,\tau) \to \theta(\vec r,\tau) + \varphi(\vec r,\tau) \\ \phi \to \phi - \partial_\tau\varphi \\ \vec A \to \vec A + \vec\nabla\varphi \end{cases} \tag{3}$$

1-3) would be satisfied by $\boxed{S[\theta] = \int d\tau d^d r \left[c_1(\partial_\tau\theta + \phi)^2 + c_2(\vec\nabla\theta - \vec A)^2\right]}$ (4)

4) requires to add also

Couplings c_1, c_2 are not fixed by above arguments.

BCS16

Phenomenological determination of c_1, c_2 (Answer: $c_1 = \partial_\mu n = \nu$, $c_2 = n_s/2m$)

Argument for c_2: Charged particles $i = 1, 2, 3 \ldots$ in vector potential have coupling term

$$S_A = \int d\tau \sum_i \dot{\vec{x}}_i \cdot \vec{A}(\vec{x}_i) = \int d\tau \int d\vec{r} \sum_i \delta(\vec{x} - \vec{x}_i)\, \dot{\vec{x}}_i \cdot \vec{A}(\vec{x}, \tau) \quad (1)$$

$$\Rightarrow \frac{\delta S_A}{\delta \vec{A}} = \sum_i \delta(\vec{x} - \vec{x}_i)\, \dot{\vec{x}}_i \propto \hat{\vec{j}}(x), \text{ so } \frac{\delta}{\delta A} \text{ generates a current density operator} \quad (2)$$

In present case QM version of argument: quantum current operator:

$$\left\langle \frac{\delta S_A}{\delta \vec{A}} \right\rangle \overset{(1.2)}{=} -\frac{1}{2m} \left\langle \bar{\psi}_\sigma(-i\vec{\nabla} - \vec{A})\psi_\sigma + \left[(i\vec{\nabla} - \vec{A})\bar{\psi}_\sigma\right]\psi_\sigma \right\rangle \equiv \langle \hat{\vec{j}} \rangle \quad (3)$$

Assume $\vec{j} = \vec{j}_n + \vec{j}_s$ (two-fluid model) (4)

normal, superconducting part of current carried by electrons with density n, n_s.

BCS17

Assume state ψ_s participating in condensate carries collective phase θ with nonvanishing average:

$$\psi_s = |\psi_s| e^{i(\theta_s + \theta)} \quad (1)$$

averages to zero — survives averaging

insert (1) into (16.3) ignore density variations:

$$\text{then } \left\langle \frac{\delta S}{\delta A} \right\rangle \overset{(16.3)}{=} -\frac{n_s}{m} \langle \vec{\nabla}\theta - \vec{A} \rangle \overset{(16.4)}{=:} \langle j_s \rangle \quad \left(\text{where } n_s = \bar{\psi}_s \psi_s = \text{condensate density}\right) \quad (2)$$

Now use (15.4) $= -2c_2(\vec{\nabla}\theta - \vec{A}) \Rightarrow \boxed{c_2 = \frac{n_s}{2m}}$ (3)

Argument for c_1:

Assume a slowly varying external potential perturbation, $\phi(\vec{r}, \tau)$, and, in response, adiabatically varying density:

single-part. density of states

$$\delta n(\vec{r}, \tau) = \delta n(\mu + \phi(\vec{r}, \tau)) \simeq \frac{\partial n}{\partial \mu} \phi(\vec{r}, \tau) \quad (4)$$

Resulting potential energy: $$\delta E_V = \int d^d x \; \phi(\bar{r},\tau)\,\delta n(\bar{r},\tau) \qquad (1)$$

$$\overset{(17.4)}{=} \partial_\mu n \int d^d x \; \phi^2(\bar{r},\tau) \qquad (2)$$

Compare to $\int d\tau \int d^d r \;\; c_1\left(\partial_\tau\theta - \phi\right)^2$, then conclude: $\boxed{c_1 = \partial_\mu n = \nu}$ (3)

<u>Microscopic derivation:</u> expand Eq (1) to lowest order in $\partial_\tau\theta$, $\partial_r\theta$, A, ϕ.

tedious but instructive! Result:

$$n_s = n + \frac{2\nu\mu}{d}\int d\xi \;\beta\; n_F(\lambda_\xi)\left[1 - n_F(\lambda_\xi)\right] \qquad (4)$$

<u>Meissner effect and Anderson-Higgs mechanism:</u>

Key feature of superconductivity is: quantum phases of macroscopically large number of particles get locked into a collective degree of freedom, described by $\Delta(\bar{r},\tau)$. Let's explore consequences: To simplify discussion:

1) assume no electric fields are present: $\phi = 0$, $\partial_\tau \vec{A} = 0$.

2) assume high temp., so "quantum fluctuations" don't contribute,

since $\int d\tau \int d^d r \; \nu\left(\partial_\tau\theta\right)^2 + S_{cl}[\theta]$

$$\overset{(17)}{=} \sum_{\bar{q},n} \underbrace{\nu\,\omega_n^2}_{\nu(2\pi T)^2 n^2}\,\theta_{n,\bar{q}}\,\theta_{-n,-\bar{q}} + S_{cl}[\theta]$$

gets more wiggles as $\beta \to 0$, $T \to \infty$

$\nu(2\pi T)^2 n^2$, so $n \neq 0$ modes get strongly suppressed as $T \to \infty$ $\Rightarrow$ only $\theta(\bar{r},\tau) = \theta(\bar{r})$ survives!

Simplify action: (4.1) & (15.4) (no time-dependence). BCS20

$$S[\vec{A},\theta] = \frac{\beta}{2}\int d^d r \left[\frac{n_s}{m}(\vec\nabla\theta - \vec A)^2 + \underbrace{(\vec\nabla\times\vec A)^2}_{\vec B^2} \right] \quad (1)$$

$\vec B^2$, from $S_{E.M.}$

- gauge invariant ✓

$$= \frac{\beta}{2}\sum_{\vec q}\left(\frac{n_s}{m} \underbrace{(i\vec q\,\theta_{\vec q} - \vec A_{\vec q})\cdot(-i\vec q\,\theta_{-\vec q} - A_{-\vec q})}_{\theta_q q^2 \theta_{-q} \; - 2i\theta_q \vec q\cdot\vec A_{-\vec q} \; + \; \vec A_q\cdot\vec A_{-q}} + (\vec q\times\vec A_{\vec q})\cdot(\vec q\times\vec A_{-\vec q}) \right) \quad (2)$$

- quadratic in θ, so integrate out θ to generate action $S[\vec A]$.

$$S[\vec A] = \frac{\beta}{2}\sum_{\vec q}\left[\frac{n_s}{m}\left(\vec A_{\vec q}\cdot\vec A_{-\vec q} - \frac{(\vec q\cdot\vec A_{\vec q})(\vec q\cdot\vec A_{-\vec q})}{q^2}\right) + (\vec q\times\vec A_{\vec q})\cdot(\vec q\times\vec A_{-\vec q})\right] \quad (3)$$

Decompose:
$$\vec A_{\vec q} = \underbrace{\vec A_{\vec q} - \frac{\vec q(\vec q\cdot\vec A_q)}{q^2}}_{\vec A_{\vec q}^{\perp}} + \underbrace{\vec q(\vec q\cdot\vec A_{\vec q})}_{\vec A_{\vec q}^{\parallel}} \quad (4)$$

with $\vec q\cdot\vec A_q^{\perp} = 0$

Then
$$\boxed{S[\vec A] \overset{\text{elementary algebra}}{=} \frac{\beta}{2}\sum_{\vec q}\left(\frac{n_s}{m} + q^2\right) A_q^{\perp}\cdot A_{-q}^{\perp}} \quad (1)$$ BCS21

(mass term!)

Reasons for this decomposition

- Only transverse component is relevant, since $\vec B_q = i\vec q\times\vec A$ (2)

  as longitudinal part gives $\vec q\times\vec q = 0$.

- $\vec A_q^{\perp}$ is invariant under $\vec A\to\vec A_q + i\vec q\,\phi_{\vec q}$ (check, insert into (20.4)) (3)

<u>What has been achieved?</u>

- $S[\vec A,\theta] \xrightarrow{\int D\theta} S[\vec A]$, so gauge degree of freedom has been "absorbed" into $\vec A$, by exploiting gauge symmetry
- $S[A]$ is gauge invariant!
- Due to coupling to Goldstone mode, gauge field $\vec A$ acquires a <u>mass</u>!

  "photon (vector potential) field has consumed Goldstone mode to become massive"

This is the "Anderson-Higgs mechanism"! $\boxed{\text{BCS22}}$

<u>Detour:</u> How do gauge bosons in standard model (SM) get a mass?

SM contains a doublet $\Psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}$, and is symmetric (1)

under $U(1) \times SU(2)$, where $SU(2)$ <u>mixes</u> ν_e, e components,

thus <u>unifying</u> EM- and weak interactions.

Action contains $\int d\tau \int d^d r \; \mathcal{L}(\bar{\Psi}, (\partial_\mu - iW_\mu)\Psi) + S[W]$, (2)

symmetric under
$$\Psi \to U \Psi \qquad \text{with } U = e^{i\phi} \in U(1) \quad \text{or } U = e^{i\vec{\phi}\cdot\vec{\sigma}} \in SU(2) \tag{3}$$

$$W_\mu \to W_\mu + i \underbrace{U^{-1}\partial_\mu U}_{\in \text{ Lie algebra generating } U} \tag{4}$$

$\boxed{\text{BCS23}}$

Now, pure gauge bosons (W, Z) are massless, with long-range propagators decaying as r^{-1}

But, experiment shows that they are <u>very</u> massive, extremely short range: $\exp(-90\,\text{GeV}\cdot r)$

Weinberg-Salam propose to "give them mass" via Higgs mechanism:

postulate a new scalar field, Φ, with action

minimal coupling provides local gauge symmetry

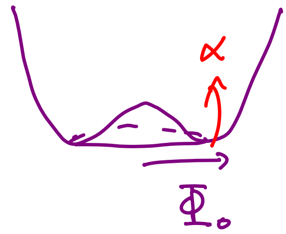
$$S[\Phi, W_\mu] = \int d^{d+1}x \left[\tfrac{1}{2} (\partial_\mu - W_\mu)\Phi^\dagger (\partial_\mu - W_\mu)\Phi - \tfrac{m}{2}\Phi^\dagger\Phi + \tfrac{g}{2}(\Phi^\dagger\Phi)^2 \right] \tag{1}$$

chosen deliberately to generate spontaneous symmetry breaking.

BCS24

for Φ-field, with $\Phi = |\Phi_0| e^{i\alpha}$,

with fixed magnitude $|\Phi| = \left(\frac{m^2}{2g}\right)^{1/2}$ and undetermined phase α.



This will generate a term of the type

$$\int d^{d+1}x \; (\Phi_0)^2 (\partial_\mu \alpha - W_\mu)(\partial_\mu \alpha - W_\mu), \qquad (1)$$

and, after integrating out $\int \mathcal{D}\alpha$,

produce a mass term for $W_\mu W^\mu$. !!

Observing the Higgs particle has been difficult so far, since it is very heavy: > 100 GeV

BCS25

Meissner effect

drop subscripts henceforth

Back to $S[A] \overset{(21.1)}{=} \frac{\beta}{2} \sum_q \left(\frac{n_s}{m} + q^2\right) A_q^\perp \cdot A_{-q}^\perp$ (1)

Saddle-point for $\int \mathcal{D}(\bar{A}, A)$: $\quad \frac{\delta S}{\delta A} = 0 \Rightarrow \left(\frac{n_s}{m} - \nabla^2\right) A(\vec{r}) = 0$ (2)

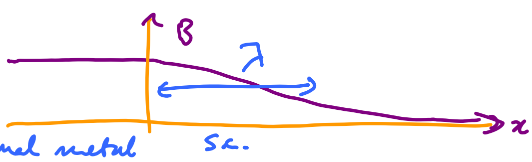
$\vec{\nabla} \times (2)$: $\quad \left(\frac{n_s}{m} - \vec{\nabla}^2\right)\vec{B} = 0$, "first London equation" (3)

has no constant solution if $n_s \neq 0$

$\Rightarrow$ bulk s.c. cannot support magnetic field $\Rightarrow$ Meissner effect!

At sc–vacuum interface; $\vec{B}(x) = \vec{B}_0 e^{-x/\lambda}$, $\lambda = \sqrt{\frac{m}{n_s}}$ (4)

= penetration length



How is magnetic field expelled? $\boxed{BCS26}$

By countercurrents that generate a field cancelling external field.

Check current:

$$\vec{j}(\vec{r}) \overset{(16.3)}{=} \frac{\delta S}{\delta \vec{A}} \overset{(25.1)}{=} \frac{\delta}{\delta \vec{A}} \int d^d r \, \frac{n_s}{2m} \vec{A}^2 = \frac{n_s}{m} \vec{A}(\vec{r}) \qquad (1)$$

(2nd London eq.!)

$\Rightarrow$ $j(\vec{r})$ is proportional to $\vec{A}(\vec{r})$, and counteracts it. (2)

Microscopic reason for Meissner effect: in magnetic field,

pair amplitude $\langle \vec{r}, \vec{r} \,|\, \vec{k}\uparrow, -\vec{k}\downarrow \rangle \sim e^{-i\int d\vec{r}\cdot(\vec{k}-e\vec{A})} e^{i\int d\vec{r}\,(-\vec{k}-e\vec{A})}$ (3)

picks up an $\vec{A}$-dependent phase, $\sim e^{2ie\vec{A}\cdot\vec{r}}$ (4)

and hence becomes "incoherent" ...

$\boxed{BCS27}$

<u>Supercurrent:</u>

Choose gauge such that $\vec{E} = -i\partial_\tau \vec{A}$. (1)

then $$-i\partial_\tau \vec{j} \overset{(31)}{=} -i\partial_\tau \frac{n_s}{m} \vec{A}^\perp \overset{(33)}{=} \frac{n_s}{m} \vec{E} \qquad (2)$$

So, in presence of field current increases linearly

$\Rightarrow$ "dissipationless current", increases without bounds ...

Actually, this is unphysical; (2) tells us that

S.C. cannot maintain nonvanishing field gradients.