Disorder IV: Low-Energy Field Theory

Goal: to recover diffusons/Cooperons from Q-field theory:

(III.12.1)
$$Z = \int \mathcal{D}Q \, e^{-\frac{\pi\nu}{8\tau}\int d^d r \, \mathrm{tr}\, Q^2 + \frac{1}{2}\mathrm{tr}\ln \hat{G}^{-1}[Q]} \tag{1}$$

(III.12.3)
$$\hat{G}^{-1}[Q] = -i\hat{\omega} + \xi_{\vec{p}} - \frac{i}{2\tau}Q \,, \tag{2}$$

MF-Equation from (1) was: (III.13.3)
$$\pi\nu\, Q(\vec{q}) + i L^{-d}\sum_{\vec{p}} \hat{G}[Q]_{\vec{p},\vec{p}-\vec{q}} = 0 \tag{3}$$

Solution: (III.13.5)
$$Q = \Lambda = \mathrm{sgn}(\omega_n) \tag{4}$$

Key observation: MF solution breaks a symmetry under $\psi \to T\psi$, $\bar{\psi} \to \bar{\psi}\bar{T}$ (to be specified below)

Construct corresponding Goldstone modes for this broken symmetry.

They will turn out to correspond to diffusons/Cooperons.

Symmetry of original action (before HS-transf. to introduce Q)

(III.3.6), (III.5.1)
$$S[\psi,\bar{\psi}] = -\frac{1}{2}\int d^d r \, d\tau \, \bar{\psi}\,\hat{G}^{-1}[V]\,\psi + \frac{1}{4\pi\nu\tau}\int d^d r \int d\tau d\tau' \, \frac{1}{2}(\bar{\psi}\psi)(\tau)\frac{1}{2}(\bar{\psi}\psi)(\tau') \tag{1}$$

Consider transf. of the type
$$\boxed{\psi \to \psi' = T\psi \,, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\bar{T}} \tag{2}$$

which respect the standard relation: (III.6.3)'
$$\bar{\psi}' \equiv -\psi'^{T}(i\sigma_2) \tag{3}$$

$$\Rightarrow \bar{\psi}\bar{T} \overset{(3)}{=} -(\psi^T T^T)\, i\sigma_2 \overset{(III.4.2)\, i\sigma_2}{=} -\bar{\psi}(i\sigma_2)\, T^T i\sigma_2 \Rightarrow \boxed{\bar{T} = -(i\sigma_2)\, T^T (i\sigma_2)} \tag{4}$$

(3) leaves (1),(2) invariant if:

- $$\bar{\psi}\psi \underset{\uparrow}{=} \bar{\psi}'\psi' \overset{(3)}{=} \bar{\psi}\bar{T}T\psi \to \bar{T}T = 1 \Rightarrow \bar{T} = T^{-1} \tag{5}$$

requirement, from $\left\{\begin{matrix}2nd\\1st\end{matrix}\right\}$ term of (1)

- $$\bar{\psi}\hat{G}^{-1}\psi \underset{\downarrow}{=} \bar{\psi}'\hat{G}^{-1}\psi' \overset{(2)}{=} \bar{\psi}\bar{T}\hat{G}^{-1}T\psi \overset{\text{see (3.1)}}{\simeq} \bar{\psi}\underbrace{\bar{T}T}_{=1}\hat{G}^{-1}\psi = \bar{\psi}\hat{G}^{-1}\psi \; ✓ \tag{6}$$

For (2.6) we used $\bar{T}\hat{g}^{-1} \simeq \hat{g}^{-1}\bar{T}$ (1) $\boxed{\text{D.III.3}}$

This holds

(i) if T is τ-independent (so that it commutes with V), and (2)

(ii) if T has only low-frequency-components, so that $\partial_\tau T \simeq 0$; (3)

in other words, in frequency space, $T_{nn'}$ should have the structure:

$$\boxed{\begin{aligned} &T_{nn'} \neq 0 \text{ only if } |n| \text{ and } |n'| < M \\ &\quad \text{i.e. if } |\omega_n|, |\omega_{n'}| < \omega_{max} = 2\pi M/\beta \end{aligned}} \tag{4}$$

Then, the only non-zero matrix elements

$$\text{of } \quad T\hat{\omega} - \hat{\omega}T = \frac{2\pi}{\beta}\begin{pmatrix} \overset{2M}{\longleftrightarrow} & \\ 2M\,\boxed{} & \\ & \end{pmatrix}\begin{pmatrix} \ddots & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & -1 \\ & & & & & -2 \\ & & & & & & \ddots \end{pmatrix} - \frac{2\pi}{\beta}\begin{pmatrix} \ddots & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & -1 \\ & & & & & -2 \\ & & & & & & \ddots \end{pmatrix}\begin{pmatrix} \overset{2M}{\longleftrightarrow} & \\ 2M\,\boxed{} & \\ & \end{pmatrix} \tag{5}$$

are of order $\omega_M = 2\pi M/\beta$ or smaller.

Symmetry group of original action $\boxed{\text{DIV.4}}$

Combining (2.4) & (2.5) gives: $T^{-1} = -(i\sigma_2)\, T^T (i\sigma_2)$ (1)

$$(i\sigma_2) = T\,(i\sigma_2)\,T^T \tag{2}$$

M: "Matsubara cutoff"

R: # of Replicas

$$\begin{pmatrix} 0 & \mathbb{1}_{2M\cdot R} \\ -\mathbb{1}_{2M\cdot R} & 0 \end{pmatrix} = T \begin{pmatrix} 0 & \mathbb{1}_{2M\cdot R} \\ -\mathbb{1}_{2M\cdot R} & 0 \end{pmatrix} T^T \tag{3}$$

(3) is the defining relation for the "unitary symplectic group" $S \equiv Sp(2MR)$

[AS call this group $Sp(4MR)$; I follow conventions of J. Cornwell, "Group Theory in Physics", 1984, Vol. 2, Table 10.1, p. 392; but this is merely a matter of convention...]

Mean field solution breaks this symmetry

The fact that T is a symmetry implies: for any solution Q of MF-equation,

$$Q \overset{(1.3)}{\sim} \hat{g} \quad (1)$$, we can generate other solutions of (1): $$Q' = T Q T^{-1} \quad (2)$$

Check: $$Q' = T Q T^{-1} \overset{(1)}{\sim} T \hat{g} T^{-1} \simeq \hat{g} \quad ✓ \quad (3)$$

Now, the solution found in (III.13.5), namely $Q_{MF} = \Lambda = \mathrm{sgn}(\omega_n)$ (4)

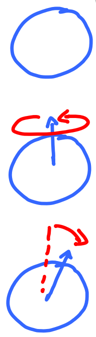
"breaks the symmetry", in the sense that it is not invariant under T:

for general $T \in S$: $$T \Lambda T^{-1} \neq \Lambda \quad (5)$$

Let $K \subset S$ denote the subgroup that does leave Λ invariant:

$$T \Lambda T^{-1} = \Lambda \quad \forall \; T \in K \subset S \quad (6)$$

Analogy: Classical magnetism



$S \to SU(2)$: rotations in 3 dimensions (1)

$K \to U(1)$: rotations around a fixed axis, say z, leave this axis invariant (2)

S/K · applied to z-axis generates Goldstone modes (3)

(magnons: oscillations in spins around z-axis).

Elements $T \in K$ are generated by $T = e^V$, with $V\Lambda - \Lambda V = 0$ (4)

time-reversal
Matsubara
replica

Now: $$\Lambda \overset{(5.4)}{=} \begin{pmatrix} \mathbb{1}_{2MR} & 0 \\ 0 & -\mathbb{1}_{2MR} \end{pmatrix}_{\text{Mats}} \quad (3)$$ where blocks refer to $\omega_n \gtrless 0$ in Matsubara space. (5)

Hence $K = Sp(MR) \times Sp(M.R) \Rightarrow K \ni T = \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}$ (6)

does not mix positive and negative freq.

Goldstone modes (GM)

Goldstone modes $\boxed{Q_{GM} = T \Lambda T^{-1}}$ (1)

"generator"

are generated from Λ by those transf. $T \in S/K$, of the form $\boxed{T = e^{W}}$, (2)

that do not commute with Λ
[i.e. "opposite" case of (6.4)]: $W\Lambda + \Lambda W = 0$ (3)

Moreover, to ensure $Q_{GM}^{\dagger} \overset{(III.8.1)}{=} Q_{GM}$, } we need: $W^{\dagger} = -W$ (4)

$e^{-W^{\dagger}} \Lambda e^{W^{\dagger}} = e^{W} \Lambda e^{-W}$

In the representation of (6.5),
(3),(4) are satisfied by: $\boxed{W = \begin{pmatrix} 0 & B \\ -B^{\dagger} & 0 \end{pmatrix}_{Mats}}$ $2MR \times 2MR$ (5)

[Check: $W\Lambda + \Lambda W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & B \\ -B^{\dagger} & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ -B^{\dagger} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$ ✓]

Non-linear constraint

$$Q_{GM}^2 \overset{(7.1)}{=} (T\Lambda T^{-1})(T\Lambda T^{-1}) = T\Lambda^2 T^{-1} = T T^{-1} = 1 \quad (1)$$

(5.4) $\hookrightarrow [\mathrm{sgn}(\omega_n)]^2 = 1$

$$\Rightarrow \quad \boxed{Q_{GM}^2 = 1} \quad (2)$$

This nonlinear constraint must be respected during $\int \mathcal{D}Q$ over symmetry-related configurations.

Field-theories with non-linear constraints are called "non-linear σ-model"

Low-Energy Field Theory

D.IV.9

Under the conditions (3.2) : T is $\vec{r}$-independent,

and (3.3) : T has only low-frequency-components,

the Goldstone modes $Q_{GM} = T \Lambda T^{-1}$ of (7.1),

subscript will be dropped henceforth.

with $T \in S/K = Sp(2MR)/Sp(MR) \times Sp(MR)$, satisfy MF eq. (1.3) : $Q_{MF} \sim \hat{g}$.

To find their dynamical behavior, now allow them to slightly violate (3.2), (3.3).

We need an effective field theory for such fluctuations, with action $S_{fl}[Q]$.

To this end,

either • substitute $Q = \{ Q^{\alpha\alpha'}_{nn'}(\vec{r}) \}$ into (1.1) and expand in
(a) gradients $\vec{\nabla} Q$, and (b) powers of $[T, \hat{\omega}] \neq 0$;

or • make informed guess based on symmetry-arguments.

Symmetry-based "Construction" of $S[Q]$, to lowest order in fluctuations

D.IV.10

Spacial dependence:

for $\vec{r}$-independent Q, action is invariant under $Q \to \tilde{T} Q \tilde{T}^{-1}$, $\tilde{T} \in S$.

thus, penalty for spatial fluctuations in Q must be of the form

$$\Rightarrow \quad S_{fl}[Q] = c_{fl} \int d^d r \; \mathrm{tr} (\nabla Q)^2 \tag{1}$$

Frequency-dependence:

$\hat{\omega}$ in $\hat{g}^{-1}$ of (1.2) breaks invariance: $[\hat{\omega}, T] \overset{(3.5)}{\neq} 0$.

function of Q

To lowest order in $\hat{\omega}$, we'll have $S_\omega[Q] \sim \int d^d r \; \mathrm{tr}(\hat{\omega} F(Q))$ (2)

(since $Q^2 = 1$, only linear terms contribute) $\overset{(8.2)}{=} [c_0 + c_\omega Q]$

$$\Rightarrow \quad S_\omega[Q] = c_\omega \int d^d r \; \mathrm{tr}(\hat{\omega} Q) \tag{3}$$

Effective action in terms of W

$$\boxed{S[Q] \overset{(10.1)+(10.3)}{=} \int d^d r \, \mathrm{tr}\left[c_{f1} (\nabla Q)^2 + c_\omega \hat{\omega} Q \right]} \quad (1)$$

[Since $Q = e^W \Lambda e^{-W}$, this is a very complicated functional of W]

To determine unknown coefficients c_{f1}, c_ω, expand in powers of B, and compare to diagrammatic results for diffuson & Cooperon:

$$Q \overset{(7.1,7.2)}{=} e^W \Lambda e^{-W} \overset{(7.3)\ [W,\Lambda]_+ = 0}{=} e^{2W} \Lambda = (1 + 2W + 2W^2 + \dots)\Lambda \quad (2)$$

$$\mathrm{tr}(\nabla Q)^2 = \mathrm{tr}\left[(2\nabla W \Lambda)(2\nabla W \Lambda) \right] = -4\, \mathrm{tr}\left[\nabla W \overset{1}{\overset{\shortparallel}{\Lambda^2}} \nabla W \right] = -4 (\nabla W)^2 \quad (3)$$

$$\mathrm{tr}(\hat{\omega} W \Lambda) = -\mathrm{tr}(\hat{\omega} \Lambda W) = -\mathrm{tr}(\hat{\omega} W \Lambda) \Rightarrow = 0. \quad (4)$$

trace is cyclic

$$\rightarrow \quad \boxed{S[Q] = \int d^d r \, \mathrm{tr}\left[-4 c_{f1} (\nabla W)^2 + 2 c_\omega \hat{\omega} W^2 \Lambda \right]} \quad (5)$$

Effective action in terms of B :

Set $W \overset{(7.5)}{=} \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix}$ where $\left.\begin{matrix} B_{nn'} \\ B^\dagger_{n'n} \end{matrix}\right\} \neq 0$ for $n > 0, n' < 0$ (1)

$$W^2 = \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} = -\begin{pmatrix} BB^\dagger & 0 \\ 0 & B^\dagger B \end{pmatrix} \quad (2)$$

$$\mathrm{tr}\, \nabla B \nabla B^\dagger = \mathrm{tr}\, \nabla B^\dagger \nabla B = \sum_{nn'}{}' (\partial_i B_{nn'})(\partial_i B^\dagger_{n'n}) \quad \text{where} \quad \sum_{nn'}{}' = \sum_{n>0} \sum_{n'<0} \quad (3)$$

make trace explicit

$$\mathrm{tr}(\hat{\omega} W^2 \Lambda) = -\mathrm{tr}\, \hat{\omega} \begin{pmatrix} BB^\dagger & 0 \\ 0 & B^\dagger B \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = -\sum_{nn'}{}' B_{nn'} (\omega_n - \omega_{n'}) B^\dagger_{n'n} \quad (4)$$

$$S[B,B^\dagger] \overset{(17.5)}{=} \sum_{\vec{q}} \sum_{nn'}{}' \sum_{\alpha\alpha'} B^{\alpha\alpha'}_{\vec{q},nn'} \Big[\underbrace{8 c_{f1}}_{\equiv cD} \overset{(3)}{q^2} - \underbrace{2 c_\omega}_{\equiv(-c)} \overset{(4)}{(\omega_n - \omega_{n'})} \Big] B^{\dagger\, \alpha'\alpha}_{-\vec{q}\, n'n} \quad (5)$$

(to make analogy to diff./Cooperon)

$$\boxed{S[B,B^\dagger] = c \sum_{\vec{q}} \sum_{nn'}{}' \sum_{\alpha\alpha'} B^{\alpha\alpha'}_{\vec{q},nn'} \left[D q^2 + |\omega_n - \omega_{n'}| \right] B^{\dagger\, \alpha'\alpha}_{-\vec{q},n'n}} \quad (6)$$

[diffusive structure emerges !!]

Interpretation of B-fields

D.IV.13

Recall $Q^{\sigma\sigma'} \sim \Psi^{\sigma} \bar{\Psi}^{\sigma'}$, where $\Psi_\tau \equiv \begin{pmatrix} \psi_\tau \\ \bar{\psi}^T_{-\tau} \end{pmatrix}$, $\bar{\Psi} \equiv \left(\bar{\psi}_\tau , -\psi^T_{-\tau} \right)$

time-reversal indices of (III.4.1)

So:

$$\begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix} \sim \begin{pmatrix} \psi_\tau \bar{\psi}_\tau & -\psi_\tau \psi^T_{-\tau} \\ \bar{\psi}^T_{-\tau} \bar{\psi}_\tau & -\bar{\psi}^T_{-\tau} \psi^T_{-\tau} \end{pmatrix} \sim \begin{pmatrix} \text{diffuson} & \text{Cooperon} \\ \text{Cooperon} & \text{diffuson} \end{pmatrix}$$

B diagonal: diffuson modes ; B off-diagonal: Cooperon modes

In absence of magnetic field, action (12.6) of $B^{\sigma\sigma}$ and $B^{\sigma\bar{\sigma}}$ are identical; <u>with</u> magnetic field, they will differ...

Determining coefficient c

D.IV 14

Introduce a source term $S[\psi,\bar{\psi}] \to S[\psi,\bar{\psi}] - \frac{1}{2L^d} \sum_{\bar{p},n} \bar{\Psi}_{\bar{p},n} K_n \Psi_{\bar{p}n}$ (1)

such that differentiation w.r.t. K_n reproduce density of states, as follows:

(D.I.7.1) gives factor 2 for t.r.

$$\lim_{R\to 0} \frac{1}{R} \mathrm{Im} \frac{\delta}{\delta K_n} \Big|_{K=0} Z^R = \frac{1}{2L^d} \mathrm{Im} \sum_{\bar{p}} \underbrace{\lim_{R\to 0} \frac{1}{R} \sum_a}_{\text{use just one replica.}} \sum_\sigma \langle \bar{\psi}^{a\sigma}_{\bar{p},n} \psi^{a\sigma}_{\bar{p},n} \rangle_\psi \quad (2)$$

$$= \frac{1}{L^d} \mathrm{Im} \sum_{\bar{p}} G_{\bar{p},n} \overset{\text{see (D.II.4.1)}}{=} -\nu\pi \, \mathrm{sgn}(\omega_n)$$ DOS shows up here! (3)

D.IV.15

Now, how does source affect $S_\omega[Q]$ of (10.3) ?

Adding source in (14.1) means $-i\hat{\omega} \to -i\hat{\omega} - \frac{K_n}{L^d}$ (1)

So, $S_\omega[Q] \xrightarrow{(10.3)} c_\omega \int d^q_r \, \mathrm{tr}\left[\left(\hat{\omega} - i\frac{K_n}{L^d}\right)Q\right]$. (2)

$$-\nu\pi \, \mathrm{sgn}(\omega_n) \overset{(14.3)}{=} \lim_{R\to 0}\frac{1}{R}\, \mathrm{Im} \frac{\delta}{\delta K_n}\Big|_{K=0} Z^R = \lim_{R\to 0}\frac{1}{R}\frac{c_\omega}{L^d}\int d^q_r \, \underbrace{\mathrm{tr}\, Q_{nn}}_{\sum_{a=1}^{R}\sum_{\sigma=1}^{2}\left(\underbrace{\Lambda^{a\sigma,a\sigma}_{nn}}_{\mathrm{sgn}(n)}+\dots\right)} \tag{3}$$

(11.2) $\Lambda_{nn} + \dots$

$$= 2\, c_\omega \, \mathrm{sgn}(\omega_n) \tag{4}$$

$$\Rightarrow \quad c_\omega \overset{(4)}{=} -\frac{\nu\pi}{2} \overset{(12.5)}{\equiv} -\frac{c}{2}, \quad \Rightarrow \quad c = \nu\pi \quad \Rightarrow \quad c_{fl} \overset{(12.5)}{=} \frac{cD}{8} = \frac{\nu\pi D}{8} \tag{4}$$

$$\boxed{S[Q] \overset{(11.5)}{=} \frac{\pi\nu}{2}\int d^q_r \, \mathrm{tr}\left[\frac{D}{4}(\nabla Q)^2 - \hat{\omega} Q\right]} \tag{5}$$

NON-LINEAR
σ-MODEL

D.IV.16

Effect of external magnetic field: exploit gauge invariance:

Under gauge transformation $\psi_\tau \to e^{i\phi_\tau}\psi_\tau$ (1)

$$\Psi_\tau = \begin{pmatrix}\psi_\tau \\ \bar{\psi}^T_{-\tau}\end{pmatrix} \to \begin{bmatrix} e^{i\phi_\tau}\psi_\tau \\ e^{-i\phi_{-\tau}}\bar{\psi}^T_{-\tau}\end{bmatrix} = e^{i\Phi_\tau}\Psi_\tau, \text{ with } \Phi_\tau = \begin{pmatrix}\phi_\tau & 0 \\ 0 & -\phi_{-\tau}\end{pmatrix} \tag{2}$$

Hence

$$Q \sim \Psi\bar{\Psi} \to e^{i\Phi} Q e^{-i\Phi} . \tag{3}$$

If Φ = const in space & time, then $S[Q]$ of (15.5) is gauge-inv.

$$\vec{\nabla} Q \to e^{i\Phi}\left[\vec{\nabla}Q + i[\vec{\nabla}\Phi, Q]\right]e^{-i\Phi} \tag{4}$$

$$\hat{\omega} Q = i\partial_\tau Q \to e^{i\Phi}\left[\omega Q - [\partial_\tau \Phi, Q]\right]e^{-i\Phi} \tag{5}$$

like Φ, $\vec{A}$ and V are diagonal matrices in t.r. space since they get $\tau \to -\tau$ too, when rewriting $\int d\tau\, \bar{\psi} A \psi \to \int d\tau\, \psi^T A^T \bar{\psi}^T$

Vector & scalar potential gauge-transform in standard manner: $\vec{A} \to \vec{A} + \vec{\nabla}\Phi$, $V \to V + \partial_\tau \Phi$ (6)

So, gauge-invariant action reads: Notation: $[A, \,]Q \equiv [A, Q]$ (7)

$$\boxed{S[Q] \overset{(15.5)}{=} \frac{\pi\nu}{2}\int d^q_r \, \mathrm{tr}\left\{\frac{D}{4}\left[(\vec{\nabla} - i[A, \,])Q\right]^2 - (\hat{\omega} + [V, \,])Q\right\}} \tag{8}$$

"MINIMAL COUPLING"