**Kondo effect in quantum dots: Anderson model**

Main results of lecture 1:

Kondo Model:

\[ H = \sum_{k\sigma} \varepsilon_k c_{k\sigma}^\dagger c_{k\sigma} + J \sum_{k\sigma} \left( c_{k\sigma}^\dagger \frac{1}{2} \sigma_{\sigma\sigma'} \cdot \sigma_{\sigma'\sigma} \right) \cdot \sigma \]  

Spin-flip scattering:

enhanced at low temp:

\[ T \leq T_K = D \exp\left[\frac{-1}{k_B T}\right] \Rightarrow \text{ground state} = \text{spin singlet} \]

Scattering phase shifts at \( T = 0 \):

\[ \delta_q(\alpha) = -\delta_\alpha(\alpha) = \pi/2 \]

How do magnetic moments form in metals?
Answer provided by "Anderson impurity model" (AM) [1961], relevant also to describe transport through quantum dots, which also show Kondo effect [1998].

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**Single-impurity Anderson model**


Conduction band:
(Flat DOS, "wide-band limit": \( D \gg \text{all other scales} \))

Localized impurity level:

Hybridization:

[Usual convention: \( \varepsilon_k = \varepsilon = \text{real} \)]

Level width:
(from golden rule)

Level occupancy:

\[ n_d = \langle n_d \rangle = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\varepsilon_d}{2k_B T} \right) \]
Conductance anomalies for quantum dot in "Kondo regime"

Linear conductance:
Odd Coulomb valleys
become Kondo plateaus

Fixed gate voltage:
Anomalous T- and V-dependence

Conductance anomalies: real data

Weak Kondo effect

Goldhaber-Gordon et al., Nature 391, 156

Strong Kondo effect

van der Wiel et al., Science 289, 2105 (2000)
When does "Kondo plateau" arise?

Occurs for

\[ T \to 0 \quad (T < T_K) \]

and when

\[ \nu_d \approx 1 \quad \text{(odd)} \Rightarrow \text{localized spin} \]

Kondo model \( \Rightarrow \text{spin-flip states} !! \)

Spin-flip processes occur via virtual intermediate states.

Effective spin-flip rate:

\[ \gamma \frac{V^2}{\xi_d + U} - \frac{V^2}{\xi_d} = - \frac{U V}{\xi_d (\xi_d + U)} = : J (\gamma) \]
Schrieffer-Wolff transformation

\[ H_{AM} = H_{band} + H_{loc} + H_0 \]

\[ H_0 \approx O(v) \]

Idea: seek effective \( \tilde{H} \) in subspace of \( N_d = 1 \)

Try unitary transf.: \( \tilde{H} = e^{-A} H e^{A} \)

A has pert. exp. in \( \mathcal{V} \):

\[ A = 0 + O(v) + O(v^2) + \cdots \]

Expand \( \tilde{H} \):

\[ \tilde{H} = (H_0 + H_1) + \left[A, H_0 + H_1\right] + \frac{1}{2} \left[A, \left[A, H_0 + H_1\right]\right] \]

Demand: \( \tilde{H} \) contains no \( O(v) \):

\[ H_1 = -\left[A, H_0\right], \quad H_0 \implies \tilde{H} = H_0 + \frac{1}{2} \left[A, H_0\right] + O(v^2) \]

Check algebra yourself!

(5) is satisfied by:

\[ A = \sum_{k'\sigma} \mathcal{V} \left[ \frac{1}{\varepsilon_k - \varepsilon_{k'}} c_{k\sigma}^d d_{k'^\sigma} + \frac{U}{(\varepsilon_k - \varepsilon_{k'})^2} d_{k\sigma}^c c_{k'^\sigma}^c d_{k'^\sigma} \right] - \Delta C. \]

Effective Hamiltonian for \( n_d = 1 \) yields Kondo model

(7.5) yields:

\[ \tilde{H} = \sum_{k, \sigma} \varepsilon_k c_{k\sigma}^d c_{k\sigma}^c + \sum_{k'k} \mathcal{V}_{k'k} \left[ c_{k'\sigma}^d c_{k\sigma}^c \right] + \left( c_{k'\sigma}^c c_{k\sigma}^d \text{- terms} \right) \]

Local spin operators:

\[ \sigma^z = \frac{1}{2} \left( d_{\sigma}^c d_{\sigma}^d \right), \quad \sigma^+ = d_{\sigma}^d, \quad \sigma^- = d_{\sigma}^c \]

Conduction band spin operators:

\[ \tilde{S} = \frac{1}{2} \sum_{c, \sigma} \sigma_{c, \sigma}^+ c_{c, \sigma}^c \]

Coupling:

\[ \mathcal{V} = \frac{1}{2} \sum_{k, \sigma} \frac{U}{(\varepsilon_k - \varepsilon_{k'})^2} \left< \Delta_{\sigma\sigma} \right> \]

Low-en. properties of AM for \( n_d = 1 \) described by KM:

\[ H_{Kondo} = H_{band} + J \sum_{k, \sigma} \tilde{S}_{\sigma\sigma} \frac{c_m^d c_m^c}{\text{SU(2)} \text{- symmetric!}} \]

Eff. Kondo temp: \( T_K = D e^{-\frac{1}{k_B T_K}} = D \exp\left[ -\frac{\pi |e_d| |e_d| u}{\nu} \right] \)

Agrees with Bethe Ansatz, except for prefactor (observed: see AM 5!)
Single-level quantum dot with two leads

Two-lead Hamiltonian:
\[ H = \sum_{k\sigma} E_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{k\sigma} V_k c_{k\sigma}^\dagger d_\sigma + \text{h.c.} + H_{\text{loc}} \]  

Schrieffer-Wolff:
\[ \sum_{k} V_k \rightarrow \sum_{k} U_k \left( \begin{array}{c} \ldots \end{array} \right) \]  

Effective Hamiltonian for \( n_d = 1 \):
\[ H_{\text{Kondo}} = \sum_{k\alpha} \left( -\frac{v_{k\alpha}}{\epsilon_k (\epsilon_k + u_k)} \right) \left( \sum_{k\beta} \frac{c_{k\sigma}^\dagger c_{k\beta}}{\epsilon_k + \epsilon_\beta} \right) \]  

Determinant:
\[ \det J_{\omega_1} = \zeta_c \left( \frac{v_{d2}^2 - v_{d1}^2}{v_{d2}} \right) = 0 \quad \text{one eigenvalue = 0} \]

Diagonalization of coupling matrix \( J \)
\[ J = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \tan \theta = -\frac{v_{d2}}{v_{d1}} \]

Diagonal form:
\[ \tilde{J} = JWJ^\dagger = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \zeta_c (v_{d2}^2 + v_{d1}^2) & 0 \\ 0 & 0 \end{bmatrix} \]

Rotate basis:
\[ \psi_{k\sigma} = \frac{1}{\sqrt{2}} \begin{bmatrix} j_k & \epsilon_k \end{bmatrix} \quad \sum_{k\beta} \left( \frac{c_{k\sigma}^\dagger c_{k\beta}}{\epsilon_k + \epsilon_\beta} \right) = \begin{bmatrix} \zeta_c (v_{d2}^2 + v_{d1}^2) & 0 \\ 0 & 0 \end{bmatrix} \]

J-diagonal Kondo Hamiltonian:
\[ H = \sum_{k\sigma} E_k \psi_{k\sigma}^\dagger \psi_{k\sigma} + \tilde{J}_1 \tilde{\sigma}_i \tilde{\sigma} \]

Important conclusion: One mode yields Kondo-Hamiltonian, other mode decouples completely!

Comment: for multilevel AM, coupling matrix is more complicated:
\[ J_{22} \text{ can be nonzero, because } \det J_{\omega_1} = \sum_{j\ell} \left( \psi_j^\dagger \psi_{\ell}^2 - \psi_{j}^\dagger \psi_{\ell} \right)^2 = 0 \]
Conductance through (many-level) QD with 2 leads

Consider $T = 0$, $B > 0$ and $\omega \to 0$
which ensures non-degenerate ground state

Then incident electrons experience only potential scattering, described by 2x2 S-matrix:

$$ S_{\omega, \text{lead}}(0) = W^T D_{\omega} W, \quad D_{\omega} = \begin{bmatrix} e^{i \delta_{\omega}} & 0 \\ 0 & e^{-i \delta_{\omega}} \end{bmatrix} $$

(1)

same as in (10.1)

Phase shifts:

$$ \delta_{\omega}, \text{ with } \gamma = 1, 2, \sigma = \uparrow, \downarrow $$

Landauer formula for conductance:

$$ G(T = 0) = \frac{e^2}{h} \sum_{\sigma} \left| S_{\omega, \text{lead}}(0) \right|^2 = g_0 \sum_{\sigma} \frac{1}{2} \sin^2 (\delta_{1\sigma} - \delta_{2\sigma}) $$

(10.1)

Prefactor:

$$ g_0 = \frac{2 e^2}{h} \sin^2 \theta_0 = \frac{2 e^2}{h} \frac{\delta_{(v_L,v_R)}^2}{(\delta_{v_L}^2 + \delta_{v_R}^2)^2} = \frac{2 e^2}{h} \text{ if } v_L = v_R $$

(10.2)

Important conclusion: $T = 0$ conductance is determined purely by phase shifts!

Conductance through 1-level QD with 2 leads

For 1-level AM:

$$ \Sigma_{\omega} = 0, \quad \rightarrow \delta_{1\sigma} = 0 $$

(10.3)

From lecture 1, (K9.4):

$$ a^T T = 0, \quad \delta_{1\uparrow} = - \delta_{1\downarrow} = \frac{\pi}{2} $$

(10.4)

Conductance at $T = 0$:

$$ G(T = 0) = g_0 \frac{1}{2} \sum_{\sigma} \sin^2 \left( \delta_{1\sigma} - \delta_{2\sigma} \right) = g_0 $$

(1.23)

for symmetric couplings ($v_L = v_R$)

$$ G(v_{3T}) = \frac{2 e^2}{h} = \text{"unitarity limit", maximal possible value, as though channel were completely open!} $$

(4)

$G(T)$ vs $T$ and $T = 0$

Kondo plateau

$G(v_{3T})$ vs $v_{3T}$

Kondo regime, with $U_d = 1$
Kondo-Abrikosov-Suhl resonance in local spectral function

Local Green's function:

\[
G^R_{d\sigma}(\omega) = \int_0^\infty dt \, e^{-i\omega t} \langle \xi d_{\sigma}(t), d_{\sigma}(0) \rangle
\]

Spectral function = local density of states (LDOS):

\[
A_{d\sigma}(\omega) = -\frac{1}{\pi} \text{Im} G^R_{d\sigma}(\omega) = \gamma_{\text{loc}}(\omega) = \rho(\omega)
\]

For \( T < T_k \), LDOS develops Kondo resonance...
which is observed directly in V-dep. of \( G \) (see AM4)

Numerical Renormalization Group calculations by Michael Sindel, 2004