String field theory tutorial

Exercise Sheet 4:
Vertices, propagators and off-shell amplitudes

1 From string theory amplitudes to vertices and propagators

Recall that the closed string theory four-point tree level amplitude is defined to be

$$\langle \tilde{c} \tilde{c} V_1(\infty) \tilde{c} \tilde{c} V_2(1) \tilde{c} \tilde{c} V_3(0) \int d^2z V_4(z) \rangle$$

with some vertex operator $V$ (not necessarily describing the tachyon) of conformal weight $(1,1)$. We already fixed three positions to be $(0,1,\infty)$. Note that for a general operator $O$ of weight $(h,h)$ the limit $z \to \infty$ is defined via

$$O(\infty) = \lim_{z \to 0} \frac{1}{|z|^2} O(-\frac{1}{z}).$$

For certain momenta the integral in (1) diverges when $V_4(z)$ gets close to one of the other punctures. We want to rewrite the integral in a way such that this divergence does not longer occur. This means we want to find an expression for the integral in those regions which is equal to the integral for momenta leading to a convergent limit.

We study the limit $z \to 0$. We first reparametrize the integral $z = yq$ with $y$ fixed. Show that this leads to

$$\int_0^{l_0} \frac{d^2q}{|q|^2} \langle \tilde{c} \tilde{c} V_1(\infty) \tilde{c} \tilde{c} V_2(1) |qy|^2 V_4(qy) \tilde{c} \tilde{c} V_3(0) \rangle$$

with $l_0$ the radius of the disc we are integrating over and small enough so that $V_4$ is still close to $V_3$.

The four operators appear in an asymmetric way in the amplitude. Three of them are multiplied by $\tilde{c} \tilde{c}$ while the fourth one is not. We want to introduce $\tilde{c} \tilde{c}$ also for that operator by introducing an anti-ghost. For that recall that the OPE of $b$ and $c$ is

$$b(w)c(z) = \frac{1}{w-z} + \text{reg.}$$

and similarly for $\tilde{b}\tilde{c}$. Use this to show that we can write

$$|z|^2 V_4(z) = \oint dw(b(w)) \oint d\tilde{w}(\tilde{b}(\tilde{w})) c\tilde{c} V_4(z) =: \{b_0, \{\tilde{b}_0, c\tilde{c} V_4(z)\}\},$$

where the contour runs around $z$ such that it does not include any other operator insertion.\(^1\) Plug this into (3) to find

$$\int_0^{l_0} \frac{d^2q}{|q|^2} \langle \tilde{c} \tilde{c} V_1(\infty) \tilde{c} \tilde{c} V_2(1) b_0 \tilde{b}_0 c\tilde{c} V_4(qy) \tilde{c} \tilde{c} V_3(0) \rangle.$$  

We can agree that we define our basic vertex operator to be $\mathcal{O}_i = \tilde{c} \tilde{c} V_i$. It turns out that the string field describing on shell physical states in closed string field theory is indeed of the form $\Psi = \sum_i \phi^i \mathcal{O}_i(0) |0\rangle$, at least in a certain gauge. You can check that $\{Q, \mathcal{O}_i\} = 0$, so it satisfies the free equation of motion.

We still have the problem of $\mathcal{O}_4$ colliding with $\mathcal{O}_3$. We can use scaling symmetry to make $\mathcal{O}_4$ $q$-independent:

$$\mathcal{O}_4(qy) = q^{L_0} \tilde{q}^{-L_0} \mathcal{O}_4(y) q^{-L_0} \tilde{q}^{L_0}.$$  

\(^1\)We normalize contour integrals by a factor of $\frac{1}{2\pi i}$. 
Show that (6) turns into
\[
\int_0^{l_o} \frac{d^2 q}{|q|^2} \langle O_1(\infty)O_2(1) b_0 \delta_0 q^L_0 \bar{q}^L_0 O_4(y)O_4(0) \rangle .
\] (8)

We are almost there. We can split our result in to different contributions by inserting a complete set of states. We normalize our states according to the bilinear product from closed string field theory:
\[
\langle O_1, O_2 \rangle = \langle O_1(\infty) c_0^+ O_2(0) \rangle ,
\] (9)

For every operator \( O_i \) we define its conjugate \( O_i^c \) via the relation
\[
\langle O_i^c(\infty), O_j(0) \rangle = \delta_{ij} .
\] (10)

The identity operator can then be written as
\[
\mathbb{1} = \sum_i O_i(0) \langle 0 | O_i(\infty) c_0^- = \sum_i c_0^+ O_i(0) \langle 0 | O_i(\infty) .
\] (11)

The amplitude is now
\[
\sum_{i,j} \langle O_1(\infty)O_2(1)O_i(0)\rangle \int_0^{l_o} \frac{d^2 q}{|q|^2} \langle O_i^c(\infty)c_0^- b_0 \delta_0 q^L_0 \bar{q}^L_0 O_j(0) \rangle \langle O_j^c(\infty)O_4(y)O_4(0) \rangle .
\] (12)

This looks like two three point functions linked by a propagator:
\[
P_{ij} = \int_0^{l_o} \frac{d^2 q}{|q|^2} \langle O_i^c(\infty)c_0^- b_0 \delta_0 q^L_0 \bar{q}^L_0 O_j(0) \rangle
\] (13)

To make this precise we rewrite the \( q \) integration: \( q = \exp (-s - i\theta) \). Show that
\[
\int_0^{l_o} \frac{d^2 q}{|q|^2} e^{-s-L_o} = \int_{s_0}^{\infty} ds \int_0^{2\pi} d\theta e^{-s(L_o+L_0)} e^{-i\theta(L_0-L_o)} = \frac{2\pi}{L_o + L_0} e^{-s_0(L_o+L_0)} \delta_{L_o,L_0}
\] (14)

for \( L_0 + L_o \) positive. Note that the right hand side is only ill defined for \( L_0 + L_0 = 0 \). So we actually found a way to write the integral in a way which is valid in greater generality.

We choose our basis to be eigenstates of \( L_0 \) and \( L_o \) with eigenvalues \( L_0 = L_o = \frac{k^2}{4} + \frac{m^2}{4} \). The propagator (13) now reads
\[
P_{ij} = \frac{1}{k^2 + m^2} e^{-s_0(k^2+m^2)} A_{ij}
\] (15)

with some matrix \( A_{ij} \).

Which channel (s,t,u) is represented with this limit? Which limit corresponds to the other two diagrams? In fact in closed string theory the whole amplitude cannot be written as a sum of the three limits (no matter how we choose \( s_0 \) in each case). It follows that we actually need a fundamental four-point vertex in closed string field theory, which covers the remaining region. However this vertex has no problems with colliding operators, because those regions are covered by the three point vertices + propagators. For higher interactions we also have to include fundamental vertices with an arbitrary number of punctures and arbitrary genus.

The story is different in bosonic open string theory. There a single three string interaction suffices to build all amplitudes.

2 Off-shell 3-point function

The tree level 3-point function is defined to be
\[
A_{0,3} = \langle \prod_{i=1}^3 O_i(z_i, \bar{z}_i) \rangle \propto (z_1 - z_2)^{2(h_3-h_1-h_2)}(z_2 - z_3)^{2(h_1-h_2-h_3)}(z_3 - z_1)^{h_2-h_3-h_1} \times c.c.
\] (16)

The numbers \( h_i, \bar{h}_i \) denote the conformal weights of the operators \( O_i = c \bar{c} V_i \). This amplitude is clearly only conformally invariant if \( h_i = \bar{h}_i = 0 \) for all \( i \), so it works only on-shell, if we want to keep conformal invariance.
A way around this was explained in the lecture. We define a general operator $O_i$ on the disc with coordinates $w_i$. We then embed this disc into our Riemann surface with coordinate $z$ using the relation $z = f_i(w_i)$ with $z_i = f_i(0)$. Under this transformation a primary operator transforms as

$$f_i \circ O_i(w) = (f'_i(w))^h_i \mathcal{F}_i(w)^h_i O_i(f_i(w))$$

(17)

The 3-point off-shell amplitude is then defined to be

$$A_{0,3} = \langle \prod_{i=1}^3 f_i \circ V_i(0) \rangle = \prod_{i=1}^3 f'_i(0)^h_i \mathcal{F}_i(0)^h_i \langle \prod_{i=1}^3 V_i(f_i(0)) \rangle.$$  

(18)

Show that with this definition, the amplitude is invariant under conformal transformations:

$$f_i \mapsto af_i + b$$  

(19)

### 3 Gauge transformations of massless states

In open string theory a general state at the first excited level can be written as

$$|\Psi\rangle = (e \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1}) |p, \downarrow\rangle.$$  

(20)

This state as a 28 free parameters. For convenience we recall that $|p, \downarrow\rangle$ satisfies

$$\alpha_0^\mu |p, \downarrow\rangle = p^\mu |p, \downarrow\rangle,$$

$$\alpha_n^\mu |p, \downarrow\rangle = c_n |p, \downarrow\rangle = b_n |p, \downarrow\rangle = b_0 |p, \downarrow\rangle = 0 \quad \forall n > 0.$$  

(21)

We want to determine the BRST invariant states as well as the gauge transformations. For this we expand the BRST operator as

$$Q = Q_0 + Q_1 + \ldots, \quad Q_0 = c_0(L_{-1}^m - 1), \quad Q_1 = c_1 L_{-1}^m + c_{-1} L_1^m + c_0 (b_{-1} c_1 + c_{-1} b_1).$$

(22)

The $Q_n$ for $n \geq 2$ will act trivially on $|\Psi\rangle$. The $L_n^m$ denote the matter Virasoro generators. Show that $Q |\Psi\rangle = 0$ demands that $p^2 = 0$ and $\beta = 0$.

Next derive the the most general exact state $|\chi\rangle$ state satisfying the Siegel condition $b_0 |\chi\rangle = 0$. Show that this allows us to set $\gamma = 0$ as well as shift $e_\mu$ by a longitudinal vector $\lambda p_\mu$. We are left with 24 degrees of freedom, as we expect from a massless vector.

The same procedure can be applied to closed strings using $Q + \mathcal{Q}$, where $\mathcal{Q}$ is the right moving version of $Q$.

### General information

The **lecture** takes place on
- **Monday** from 14:00 to 16:00 c.t. in B101 (Theresienstraße 39) and  
- **Friday** from 10:00 to 12:00 c.t. in A449 (Theresienstraße 37).

The **tutorial** takes place on
- **Tuesday** from 12:30 to 14:00 s.t. in A318 (Theresienstraße 37).

The **webpage** for the lecture and the exercise can be found at:

https://www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/sft_ws_17_18/index.html